Moduli Spaces of Linkages and Arrangements

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Abstract

We prove realizability theorems for vector-valued polynomial mappings, real-algebraic sets and compact smooth manifolds by moduli spaces of planar linkages and arrangements of lines in the projective plane.

1 Introduction

In this paper we describe the results of our papers [KM6] and [KM8]. Both papers deal with moduli spaces of elementary geometric objects, the first with arrangements of lines in the projective plane, the second with linkages in the Euclidean plane. We conclude the paper with a brief sketch from [KM6] of how the study of arrangements of lines leads to examples of Artin and Shephard groups which are not fundamental groups of smooth (not necessarily compact) complex algebraic varieties (Theorem 14.1). The problem of deciding which finitely-presented groups are the fundamental groups of smooth complex algebraic varieties is called "Serre's problem" in [Mo]. Our contribution to this problem is based on our discovery of the connection between configuration spaces of elementary geometric objects and representation varieties of Coxeter, Shephard and Artin groups, developed in [KM2]–[KM3], [KM5]–[KM6]. The reader may also find our works on polygonal linkages [KM1] (in $\mathbb{R}^2$), [KM4] (in $\mathbb{R}^3$) and [KM7] (in $\mathbb{S}^2$) to be of interest.

We devote most of this paper to our most recent work [KM8], dealing with moduli spaces of planar linkages. A *linkage* (L, ℓ) is a graph L with a positive real number $\ell(e)$ assigned to each edge e. We assume that we have chosen a distinguished oriented edge $e^* = [v_1 v_2]$ in L, we refer to $\mathcal{L} := (L, \ell, e^*)$ as *based linkage*. The *moduli space* $\mathcal{M}(\mathcal{L})$ of planar realizations of $\mathcal{L}$ is the set of maps ϕ from the vertex set of L into the Euclidean plane $\mathbb{R}^2$ (which will be identified with the complex plane $\mathbb{C}$) such that

- $\|\phi(v) - \phi(w)\|^2 = \ell([vw])^2$ for each edge $[vw]$ of L.
- $\phi(v_1) = (0, 0)$.
- $\phi(v_2) = (\ell(e^*), 0)$.

Clearly these conditions give $\mathcal{M}(\mathcal{L})$ natural structure of a real-algebraic set in $\mathbb{R}^{2r}$ where r is the number of vertices in L. The "double pendulum" (Figure 1) is a based linkage, its moduli space is the 2-torus $\mathbb{S}^1 \times \mathbb{S}^1$.

In Definition 3.7 we define *functional linkages*. They come equipped with two sets of vertices: the *inputs* $(P_1, ..., P_m)$ and the *outputs* $(Q_1, ..., Q_n)$. These vertices determine the input and output projections p, q from $\mathcal{M}(\mathcal{L})$ to $\mathbb{A}^m, \mathbb{A}^n$ so that $q \circ p^{-1}$ is the restriction of

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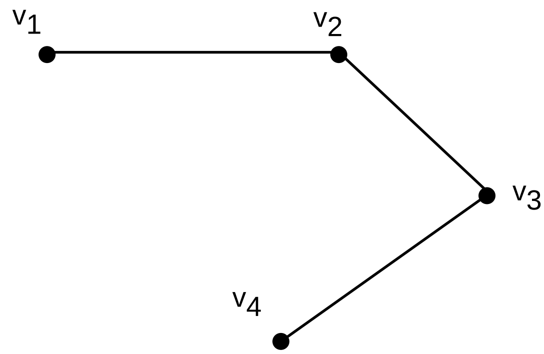

Figure 1: *The double pendulum.*

a certain polynomial mapping $f : \mathbb{A}^m \to \mathbb{A}^n$ to a domain in $\mathbb{A}^m$. Informally speaking, as the images of the input vertices $\phi(P_i)$ vary freely in a domain $Dom(\mathcal{L}) \subset \mathbb{A}^m$, the images of the output vertices are related to $(\phi(P_1), ..., \phi(P_m))$ via the function f.

Here the affine line $\mathbb{A}$ is either $\mathbb{C} \cong \mathbb{R}^2$ (in which case we refer to $\mathcal{L}$ as a *complex functional linkage*) or $\mathbb{R} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ (in which case we refer to $\mathcal{L}$ as a *real functional linkage*). Thus, for each real functional linkage the images of the input vertices are restricted to the x-axis in $\mathbb{R}^2$.

Theorem A. *Let $f : \mathbb{A}^m \to \mathbb{A}^n$ be a polynomial map with real coefficients, $B_r(\mathcal{O})$ be a ball in $\mathbb{A}^m$. Then there is a functional linkage L for f such that the input projection p is an analytically trivial algebraic covering over $B_r(\mathcal{O})$.*

We first prove Theorem A for germs at the origin 0 and then we the "expand the domain" using Theorem 7.2 to prove the general statement.

Let $S \subset \mathbb{R}^n$ be a compact real-algebraic set, i.e. it is the zero set of a polynomial $f : \mathbb{R}^n \to \mathbb{R}$. We may assume $S \subset B_r(\mathcal{O})$. We then apply Theorem A and construct a functional linkage $\mathcal{L}$ for the polynomial f. We let $\mathcal{L}_0$ be the linkage obtained from $\mathcal{L}$ by gluing the output vertex to the base-vertex v_0. We let p_0 denote the restriction of the input mapping to $\mathcal{M}(\mathcal{L}_0)$. It is shown in [KM8] that $p_0 : \mathcal{M}(\mathcal{L}_0) \to S$ is an analytically trivial algebraic covering over S. We obtain

Theorem B. *Let S be any compact real-algebraic subset of $\mathbb{R}^m$. Then there is a linkage $\mathcal{L}_0$ and an analytically trivial covering $\mathcal{M}(\mathcal{L}_0) \to S$.*

Now let M be a compact smooth manifold. By work of Seifert, Nash, Palais and Tognoli (see [AK] and [T]) M is diffeomorphic to a real algebraic set S, hence as a corollary of Theorem B we get

Corollary C. *Let M be a smooth compact manifold. Then there is a linkage $\mathcal{L}_0$ whose moduli space is diffeomorphic to disjoint union of a number of copies of M.*

We next study the analogues of Theorems A, B and Corollary C for planar arrangements. We define an arrangement $\mathcal{A}$ to be a bipartite graph with the set of vertices $\mathcal{P} \cup \mathcal{L}$ (vertices in $\mathcal{P}$ are called *points* and vertices in $\mathcal{L}$ are called *lines*). A point-vertex is said to be *incident* to a line-vertex if they are connected by an edge. We then define the projective scheme $R(\mathcal{A})$ of *projective realizations* of $\mathcal{A}$, where projective realization is a map on the set of vertices of $\mathcal{A}$ which sends the "points" of $\mathcal{A}$ to points in $\mathbb{P}^2$ and the "lines" in $\mathcal{A}$ to lines in $\mathbb{P}^2$ and preserves the incidence relation.

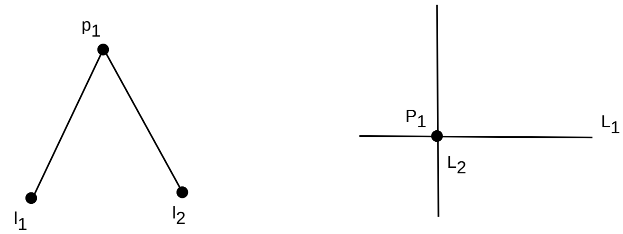

Figure 2: *Example of an arrangement $\mathcal{A}$ and its projective realization.*

For instance, the realization scheme $R(\mathcal{A})$ of the arrangement $\mathcal{A}$ described in Figure 2, is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$.

We take a cross-section, the space of *based arrangements* $BR(\mathcal{A})$, to the action of PGL_3 on a certain Zariski open and dense subset of $R(\mathcal{A})$ defined over $\mathbb{Z}$. This subset is the set of stable (and semistable) points for a suitable projective embedding of $R(\mathcal{A})$, whence $BR(\mathcal{A})$ is again a projective scheme. The cross-section involves an embedding of graphs $i : T \hookrightarrow \mathcal{A}$ for a certain arrangement T which we call the *standard triangle*, see Figure 14. We restrict to projective realizations ϕ of $\mathcal{A}$ such that $\phi \circ i$ is "standard", see Figure 14. This allows us to distinguish the x-axis, y-axis, the line at infinity and the point $(1, 1)$ in the affine plane $\mathbb{A}^2$. In Definition 9.6 we define *functional arrangements*, which (similarly to functional linkages) have input points $P_1, ..., P_m$ and output points $Q_1, ..., Q_n$ whose images under realizations are related by a function f. The input points are incident to the line l_x (corresponding to the "x-axis") in $\mathcal{A}$, hence for all realizations of $\mathcal{A}$, the images of the input vertices lie on the x-axis in $\mathbb{P}^2$.

Theorem D. *Let $f : \mathbb{A}^m \to \mathbb{A}^n$ be any morphism (i.e. a vector-valued polynomial mapping with integer coefficients). Then there is a functional arrangement for f.*

By gluing output vertices to zero we obtain an arrangement $\mathcal{A}_0$ containing distinguished vertices $P_1, ..., P_m$. Hence for each realization of $\mathcal{A}_0$ the images of the input vertices satisfy the equation $f(x_1, ..., x_m) = 0$. We define a Zariski open subscheme $BR_0(\mathcal{A}_0) \subset BR(\mathcal{A}_0)$ by requiring $\phi(P_i) \in \mathbb{A}^2$. We get the induced (input) morphism $p : BR_0(\mathcal{A}_0) \to \mathbb{A}^m$.

Theorem E. *Let S be a closed subscheme of $\mathbb{A}^m$ (again over $\mathbb{Z}$). Then there exists a based arrangement $\mathcal{A}_0$ such that the input mapping $p : BR_0(\mathcal{A}_0) \to \mathbb{A}^m$ induces an isomorphism $\tau : BR_0(\mathcal{A}_0) \to S$.*

Remark 1.1 *Because of our applications to the Serre's problem we wish to keep track of the scheme structure of $BR_0(\mathcal{A}_0)$ (e.g. keep track of nilpotent elements in the coordinate ring of $BR_0(\mathcal{A}_0)$).*

Theorem F. *Let S be a compact real algebraic set defined over $\mathbb{Z}$. Then there exists a based arrangement $\mathcal{A}_0$ such that S is* **entirely isomorphic** *to a Zariski closed and open subset of $BR(\mathcal{A}_0)(\mathbb{R})$.*

We now apply the Seifert-Nash-Palais-Tognoli theorem (here we need a strengthening to the case where polynomials have integer coefficients) to obtain

Corollary G. *Let M be a compact smooth manifold. Then there exists a based arrangement $\mathcal{A}_0$ such that M is diffeomorphic to a union of Zariski components in $BR(\mathcal{A}_0)(\mathbb{R})$.*

Remark 1.2 *It seems surprising that one can prove a somewhat stronger realization theorem for arrangements than for linkages. One explanation for this is that the image of the input map of any connected functional linkage is bounded. By a theorem of Sullivan [Sul] a manifold with nonempty boundary can not be an algebraic set* [1]. *Thus* **there are no functional linkages if we require the input map to be injective and L connected**.

We need to acknowledge a long history of previous work on linkages and arrangements. In particular, a version of Theorem A for polynomial functions $\mathbb{R} \to \mathbb{R}^2$ was formulated by A. B. Kempe in 1875 [Ke], however, as far as we can tell, his proof requires corrections (due to possible degenerate configurations). Kempe's methods were also insufficient to prove Theorem B and Corollary C even if the problem of "degenerate configurations" is somehow resolved. The second obstacle in deducing Theorem B from Theorem A is that the restriction p_0 of the regular ramified covering $p : \mathcal{M}(\mathcal{L}) \to Dom(\mathcal{L})$ to $\mathcal{M}(\mathcal{L}_0)$ apriori does not have to be an analytically trivial covering:

(a) It is possible that $\mathcal{M}(\mathcal{L}_0)$ intersects the ramification locus of p;

(b) Even if p_0 is a *topologically* trivial covering it can fail to be *analytically trivial* (for instance the function $x^3 : \mathbb{R} \to \mathbb{R}$).

Both problems of degenerate configurations and reflection symmetries of linkages were neglected (or incorrectly resolved) in the previous work we have seen. Much of the previous work was not sufficiently precise. We have formulated our results in terms of algebraic varieties (schemes) associated to realizations of graphs with additional structure and the morphisms to affine space associated to distinguished vertices in these graphs. Once we formulated our results in these terms we were forced to deal with degenerate configurations and reflection symmetries of linkages.

Thurston has been lecturing on Corollary C for twenty years. Since he has not yet written up a proof we have written our own. The methods used in our proof of Theorems D, E were used by Mnev in [Mn] (in fact they too have their roots in the 19-th century [St]). However Mnev claims only existence of a piecewise-algebraic homeomorphism between $BR_0(\mathcal{A}_0)(\mathbb{R}) \times \mathbb{R}^s$ and $S \times \mathbb{R}^k$ for some s, k. As we remarked above the scheme-theoretic version is critical for our application to Serre's problem.

We would like to thank a number of people who helped us with this work. The first author is grateful to A. Vershik for a lecture on Mnev's result in 1989. The authors thank E. Bierstone, J. Carlson, R. Hain, J. Kollar, P. Millman, C. Simpson and D. Toledo for helpful conversations related to the last part of this paper and H. King, S. Lillywhite and R. Schwartz for helpful conversations about real algebraic geometry and linkages. We are also grateful to the referee for several suggestions. The first author was supported by NSF grant DMS-96-26633, the second author by NSF grant DMS-95-04193.

2 Some Real Algebraic Geometry

An affine real algebraic set $W \subset \mathbb{R}^n$ is the set of roots of a collection of polynomial functions $\mathbb{R}^n \to \mathbb{R}$ (clearly one polynomial is enough). The set W is defined over $\mathbb{Z}$ if these polynomial functions can be chosen to have integer coefficients. Suppose that $Z \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ are affine real-algebraic sets. An *entire rational* function $f : Z \to W$ is a function which is locally (near each point of Z) the quotient of polynomials. A *entire* isomorphism $f : Z \to W$ is an entire rational function which has entire rational inverse (in particular f is a homeomorphism). If there is an entire isomorphism $f : Z \to W$ we say that Z and W are *entirely isomorphic*.

[1] The Euler characteristic of the link of a boundary point is 1.

We will identify $\mathbb{R}^n$ with the affine part of $\mathbb{RP}^n$. Suppose that $X \subset \mathbb{R}^n$ is an affine real algebraic set. Then X is said to be *projectively closed* it its Zariski closure in $\mathbb{RP}^n$ equals X. Clearly each projectively closed subset must be compact (in the classical topology). It turns out that the converse is "almost true" as well:

Theorem 2.1 *(Corollary 2.5.14 of [AK]) Suppose that $X \subset \mathbb{R}^n$ is a compact affine algebraic set. Then X is entirely isomorphic to a projectively closed affine algebraic subset X' of $\mathbb{R}^n$. Moreover if X is defined over $\mathbb{Z}$ then X' is defined over $\mathbb{Z}$ as well.*

We will need the following theorem which is a modification of [AK, Corollary 2.8.6] or [T]:

Theorem 2.2 *(Seifert-Nash-Palais-Tognoli) Suppose that M is a smooth compact manifold [2]. Then M is diffeomorphic to a projectively closed real affine algebraic set S defined over $\mathbb{Z}$.*

Remark 2.3 *This theorem is stated in [AK, Corollary 2.8.6] without the assertion that S is projectively closed and defined over $\mathbb{Z}$. We are grateful to H. King for explanation how to guarantee these extra properties of S.*

We will need another definition:

Definition 2.4 *Suppose that X, Y are real algebraic sets. Then a finite* **analytically trivial** *covering $f : X \to Y$ is an analytic map such that there is a finite set F and an analytic isomorphism $h : X \to Y \times F$ so that $f = h \circ \pi_Y$, where $\pi_Y : Y \times F \to Y$ is the projection to the first factor. We say that $f : X \to Y$ is an analytically trivial* **algebraic** *covering if it is an polynomial morphism which is an analytically trivial covering whose group G of automorphisms consists of algebraic automorphisms of X. We retain the name* **analytically trivial algebraic covering** *for the restriction of such an f to a G-invariant open subset[3] of X.*

Note, that we do not claim here that X splits into Zariski components each of which is birationally isomorphic to Y.

3 Abstract Linkages and Their Configuration Spaces

In the next five sections we will describe the two main results of [KM8] concerning the realization of polynomial maps and real algebraic sets by planar linkages. We begin with several definitions.

If L is a graph then $\mathcal{V}(L)$ and $\mathcal{E}(L)$ will denote the sets of vertices and edges of L.

Definition 3.1 *A* **marked linkage** *$\mathcal{L}$ is a triple (L, ℓ, W) consisting of a graph L, an ordered subset $W \subset \mathcal{V}(L)$ and a positive function $\ell : \mathcal{E}(L) \to \mathbb{R}_+$ (a metric on L). The elements of W are called the* **fixed vertices** *of $\mathcal{L}$ and the choice of W is called* **marking**. *If W is empty then we call $\mathcal{L}$ a* **linkage**. *A special case of a marked linkage is a* **based** *linkage where W consists of two vertices v_1, v_2 connected by an edge e^*.*

In particular, if W consists of two vertices v_1, v_2 which are the end-points of an edge e^*, then (L, ℓ, W) is the based linkage as defined in the Introduction.

[2]Not necessarily connected.
[3]With respect to the classical topology.

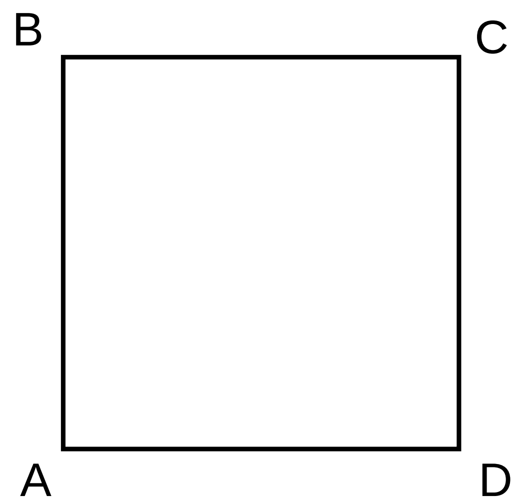

Figure 3: *The square.*

Definition 3.2 *Let $\mathcal{L} = (L, \ell, W)$ be a marked linkage. A* **planar realization** *of $\mathcal{L}$ is a map $\phi : \mathcal{E}(L) \to \mathbb{R}^2$ such that*

$$|\phi(v) - \phi(w)|^2 = \ell[vw]^2$$

for each edge $[vw]$ of L. The collection $C(\mathcal{L})$ of planar realizations of $\mathcal{L}$ is called the **configuration space** *of $\mathcal{L}$, it is clear that it has natural structure of a real-algebraic set.*

Definition 3.3 *Let $\mathcal{L} = (L, \ell, W)$ be a linkage, $W = (v_1, ..., v_n)$ be the marking. Suppose that we are given a vector $Z = (z_1, ..., z_n) \in \mathbb{C}^n$, called* **the image of marking**. *A* **relative planar realization** *of $\mathcal{L}$ is a realization $\phi \in C(\mathcal{L})$ such that $\phi(v_j) = z_j$ for all j. We let $C(\mathcal{L}, Z)$ be the set of all relative planar realizations of $\mathcal{L}$, it is called the* **relative configuration space** *of $\mathcal{L}$.*

In the case $\mathcal{L}$ is a based linkage and $Z = (0, \ell(e^*)) \in \mathbb{R}^2$, the relative configuration space equals the moduli space of $\mathcal{L}$. The algebraic set $C(\mathcal{L})$ canonically splits as the product $\mathcal{M}(\mathcal{L}) \times E(2)$ (the group $E(2)$ of orientation-preserving isometries of $\mathbb{R}^2$ has obvious real-algebraic structure), thus we shall identify the quotient $C(\mathcal{L})/E(2)$ and $\mathcal{M}(\mathcal{L})$. Note that $\mathcal{M}(\mathcal{L})$ admits an algebraic automorphism induced by the complex conjugation in $\mathbb{C} = \mathbb{R}^2$.

Many of the problems with the 19-th century work on linkages can be traced to neglecting degenerate realizations of a square. A *square* is the polygonal linkage where all four sides have equal length (see Figure 3). We have

Lemma 3.4 *The moduli space of the square is isomorphic to a union of* **three** *projective lines in general position in the projective plane.*

Proof: See [KM1, §12] and [KM4, §6].

Two of the components of the moduli space of the square consist of "degenerate" squares. We can eliminate the components consisting of degenerate squares by "rigidifying" the square as on Figure 6. We have

Lemma 3.5 *The moduli space of the rigidified square Q is isomorphic to $\mathbb{RP}^1$ (i.e. a circle). For any $\phi \in \mathcal{M}(Q)$ the points $(\phi(v_1), \phi(v_3), \phi(v_4), \phi(v_6))$ form the set of vertices of a rhombus in $\mathbb{R}^2$.*

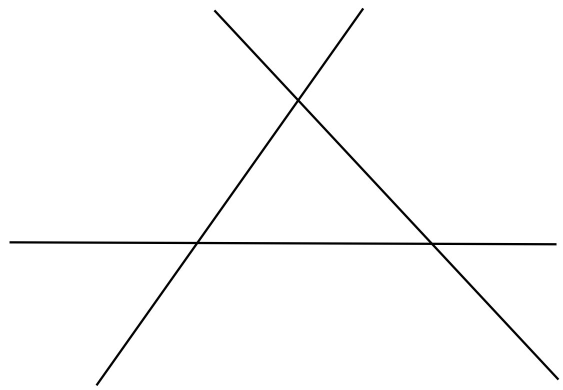

Figure 4: *The moduli space of the square.*

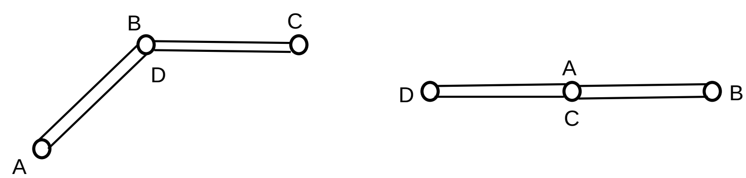

Figure 5: *Degenerate realizations of the square. Small circles denote images of the vertices.*

Remark 3.6 *In fact we have added nilpotent elements to the structure sheaf of the moduli space but we are only considering reduced structure here.*

We rigidify parallelogram linkages in an analogous way. Henceforth all parallelogram linkages that appear in this paper will be rigidified– but we will not draw the extra edges. We now give the main definition of [KM8]. Let $\mathbf{k}$ denote either $\mathbb{C}$ or $\mathbb{R}$. We will identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{R}$ with the real axis in $\mathbb{C}$.

Definition 3.7 *Let $\mathcal{O} \in \mathbf{k}^m$ and $F : \mathbf{k}^m \to \mathbf{k}^n$ be a map. We define a* $\mathbf{k}$*-functional linkage $\mathcal{L}$ for the germ $(F, \mathcal{O})$ to be a marked linkage $\mathcal{L} = (L, \ell, W)$ with m distinguished vertices $In(\mathcal{L}) = \{P_1, .., P_m\}$ (called the* **input** *vertices) and n additional distinguished vertices $Out(\mathcal{L}) = \{Q_1, ..., Q_n\}$ (called the* **output** *vertices) and a choice of the image of a marking Z satisfying the axioms:*

(1) The forgetful map $p : C(\mathcal{L}, Z) \to (\mathbb{R}^2)^m$ given by

$$p(\phi) = (\phi(P_1), ..., \phi(P_m)), \quad \phi \in C(\mathcal{L}, Z)$$

is a regular topological branched covering of a domain [4] *$Dom(\mathcal{L})$ in $\mathbf{k}^m$. We let $Dom^*(\mathcal{L})$ denote the set of regular values of $p : C(\mathcal{L}, Z) \to Dom(\mathcal{L})$. We require $\mathcal{O} \in Dom^*(\mathcal{L})$.*

(2) The forgetful map $q : C(\mathcal{L}, Z) \to \mathbb{R}^{2n}$ given by

$$q(\phi) = (\phi(Q_1), ..., \phi(Q_n)), \quad \phi \in C(\mathcal{L}, Z)$$

factors through p and induces the map $F|Dom(\mathcal{L}) : Dom(\mathcal{L}) \to \mathbf{k}^n$.

We will say that the map F is **defined** *by the linkage $\mathcal{L}$. The group of automorphisms of the branched covering p is called the* **symmetry group** *of $\mathcal{L}$. We will refer to* $\mathbb{R}$*-functional linkages as* **real functional linkages** *and* $\mathbb{C}$*-functional linkages as* **complex functional linkages**.

[4]A domain in $\mathbb{R}^N$ is a subset U with nonempty interior.

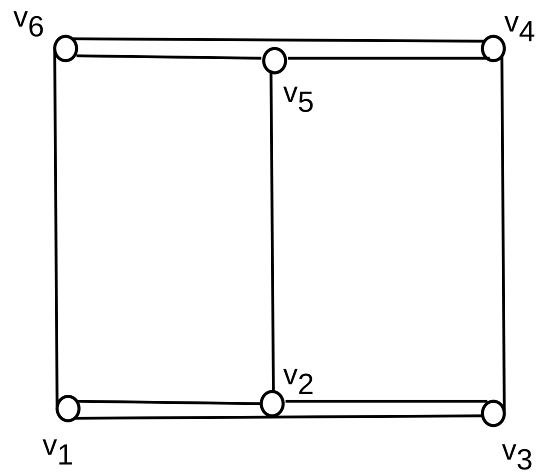

Figure 6: *The rigidified square Q. Here* $\ell[v_1v_2] = \ell[v_2v_3] = \ell[v_1v_3]/2 = \ell[v_6v_5] = \ell[v_5v_4] = \ell[v_6v_4]/2$.

It is clear that $Dom(\mathcal{L}), Dom^*(\mathcal{L})$ and the symmetry group depend also on the choice Z, we suppress this choice to simplify the notations. Notice that in the definition of a functional linkage for a germ $(F, \mathcal{O})$ the metric ball around $\mathcal{O}$ which is contained in $Dom^*(\mathcal{L})$ is not specified. We will also need the following modification of the above definition:

Definition 3.8 *Suppose that the pair $(\mathcal{L}, Z)$ as above defines the germ $(F, \mathcal{O})$ and, moreover, U is a neighborhood of $\mathcal{O}$ such that $U \subset Dom^*(\mathcal{L})$. Then we say that the pair $(\mathcal{L}, Z)$* **defines** (F, U).

Remark 3.9 *In this paper (for the sake of brevity) we will suppress the choice of $\mathcal{O}$ for certain functional linkages: it is often a tricky issue, we refer to our paper [KM8] for details. For certain (but not for all!) functional linkages the point $\mathcal{O}$ is the origin.*

Examples of functional linkages are given in section 5.

4 Fiber Sums of Linkages

The operation of fiber sum of linkages is analogous to the generalized free products of groups (i.e. the amalgamated free product and HNN-extension). Let $\mathcal{L}' = (L', \ell', W')$, $\mathcal{L}'' = (L'', \ell'', W'')$ be marked linkages. Suppose that we have a map

$$\beta : S' \to \mathcal{V}(L'')$$

where $S' \subset \mathcal{V}(L')$. If the images Z', Z'' of W', W'' are given we require

$$\phi'(w_j) = \phi''(\beta(w_j))$$

for each $w_j \in W'$ and $\phi' \in C(\mathcal{L}', Z'), \phi'' \in C(\mathcal{L}'', Z'')$. Then the *fiber sum* $\mathcal{L}$ of linkages $\mathcal{L}', \mathcal{L}''$ associated with β is constructed as follows:

Step 1. Take the disjoint union of metric graphs $(L', \ell') \sqcup (L'', \ell'')$ and identify v and $\beta(v)$ for all $v \in S'$. The result is the metric graph (L, ℓ).

Step 2. Let W be the image in L of $W' \sqcup W''$, we let W be the marking of the resulting fiber sum $\mathcal{L} := (L, \ell, W)$. If the images Z', Z'' of W', W'' are given, we define the vector Z (the image of W) as the vector with the coordinates $\phi(w_j)$, where $w_j \in W$ and ϕ is in $C(\mathcal{L}', Z')$ or in $C(\mathcal{L}'', Z'')$.

In what follows we will consider $\mathcal{L}', \mathcal{L}''$ to be canonically embedded in $\mathcal{L}$.

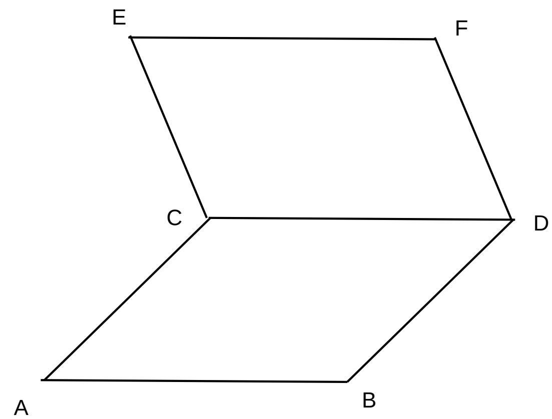

Figure 7: *A translator. The parallelograms $ACDB$ and $CEFD$ are rigidified. The set of intput/output vertices is $\{E, F\}$.*

5 Elementary Functional Linkages

The main tool in proving Theorem A is the composition operation on functional linkages, this is a fiber sum which involves identifying an output vertex (or vertices) of a functional linkage to an input vertex (or vertices) of another. Also if we have a functional linkage $\mathcal{L}_1$ for $f_1(z_1, ..., z_n)$ and a functional linkage $\mathcal{L}_2$ for $f_2(z_1, ..., z_n)$ we use fiber sum to construct a functional linkage for the vector-valued mapping $f = (f_1, f_2)$ by gluing inputs of $\mathcal{L}_1$ and $\mathcal{L}_2$. The above operations correspond to appropriate fiber products of the moduli spaces of these linkages.

Remark 5.1 *One has to show of course that such gluing again produces a functional linkage, in particular, if $\mathcal{O} \in Dom^*(\mathcal{L}_{F_1}), F_1(\mathcal{O}) \in Dom^*(\mathcal{L}_{F_2})$ then $\mathcal{O} \in Dom^*(\mathcal{L}_{F_2 \circ F_1})$. Here $\mathcal{L}_G$ is a functional linkage for polynomial vector-valued function G.*

With these observations the proof of Theorem A follows a path familiar to the researchers of the 19-th century. Because we can compose linkages the problem reduces to constructing an adder and a multiplier which is done in this section. We will also need several other auxiliary "elementary" linkages. All *elementary* linkages in the section (with the exception of the multiplier) are modifications of classical constructions, where appropriate modification was made to ensure functionality. We decided to avoid Kempe's construction of the multiplier [Ke] since the computation of Dom and Dom^* for Kempe's linkage presents some difficulties, we use an algebraic trick instead.

(1) *The translators.* Let b be a fixed positive number. The translation operations $\tau_b : z \mapsto z + b$, $\tau_{-b} : w \mapsto w - b$ are defined using the translator which is described in Figure 7. We let $W := (A, B)$ be the marking and $Z = (0, b)$ be its image. Depending on the situation either F or E is the input (resp. output). The point is that if E is the input then by adjusting side-lengths of the translator we can get any $z \in \mathbb{C} - \{0\}$ into Dom^* of this $\mathbb{C}$-functional linkage for τ_b. To get $0 \in \mathbb{C}$ into Dom^* we use the point F as the input (and E as the output) of a functional linkage $L_{\tau_{-b}}$ for τ_{-b}. If

$$\ell[BD] + \ell[DF] > b > \ell[DF] - \ell[DB] > 0$$

then the origin belongs to $Dom^*(L_{\tau_{-b}})$.

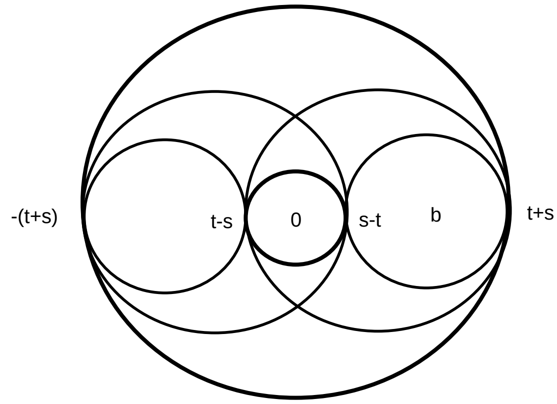

Figure 8: *Domain of the translator.*

It is clear that the relative configuration space of each translator $\mathcal{L}$ is the same as for the double pendulum, i.e. the 2-torus. The group of symmetries is $\mathbb{Z}/2$, it is generated by the transformation which fixes $\phi(E), \phi(F)$ and simultaneously reflects the points $\phi(C), \phi(D)$ in the lines $(\phi(E), O), (\phi(F), b)$. The fixed-point set of this symmetry consists of two circles and $Dom(\mathcal{L})$ is the annulus in $\mathbb{C}$. To obtain $Dom^*(\mathcal{L})$ we remove the boundary circles C_1, C_2 of this annulus as well as four other circles that are orthogonal to the real axis and tangent to C_1, C_2, see Figure 8.

The adder. This linkage is described in Figure 9. We let $W := \{v_1\}, Z := (0)$. Notice that the point $(0,0) \in \mathbb{C} \times \mathbb{C}$ does not belong to Dom^* of this linkage. To get a functional linkage for the addition in a neighborhood the origin we use the formula:

$$z + w = (z + b) + (w - b)$$

where $(b, -b)$ belongs to Dom^* of the adder. The functions $\tau_b : z \mapsto z + b, \tau_{-b} : z \mapsto z - b$ are constructed using the appropriate translators.

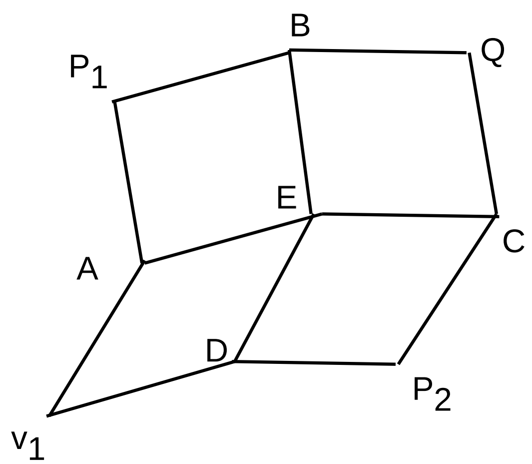

Figure 9: *The adder. The vertices P_1 and P_2 are inputs and Q is the output, the four parallelograms are rigidified. The point $\phi(v_1) = (0,0) \in \mathbb{C}^2$ does not belong to Dom^*. We use translators to resolve this problem.*

(3) The *rigidified pantograph.* This linkage is described on Figure 10.

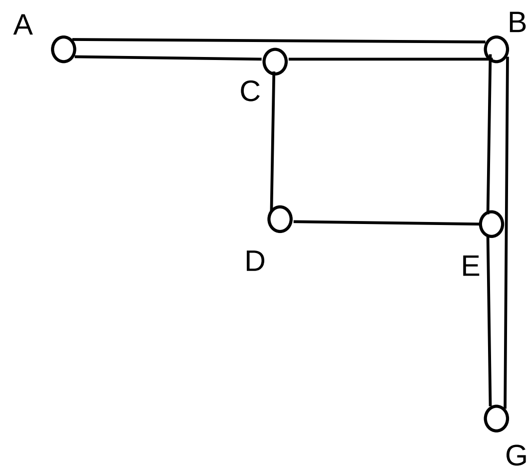

Figure 10: *The rigidified pantograph $\mathcal{P}$: the parallelogram $BCDE$ is rigidified, $\lambda > 1$. This linkage is not marked, we shall use different choices of input/output vertices later on. We take: $s = \ell[AB] = \lambda\ell[AC] \neq t = \ell[BG] = \lambda\ell[BE]$.*

The pantograph is a versatile linkage, its role in engineering was as a functional linkage for the functions $z \mapsto \lambda z, z \mapsto \lambda^{-1} z$, $\lambda > 1$. In the case of the function $z \mapsto \lambda z$ we let $W := \{A\}$ be the fixed vertex, $Z := 0$, take D as input and G as output, let $\mathcal{P}_\lambda$ be the resulting linkage (it will be functional for $z \mapsto \lambda z$). By switching input and output we obtain a functional linkage $\mathcal{P}_{1/\lambda}$ for $z \mapsto z/\lambda$.

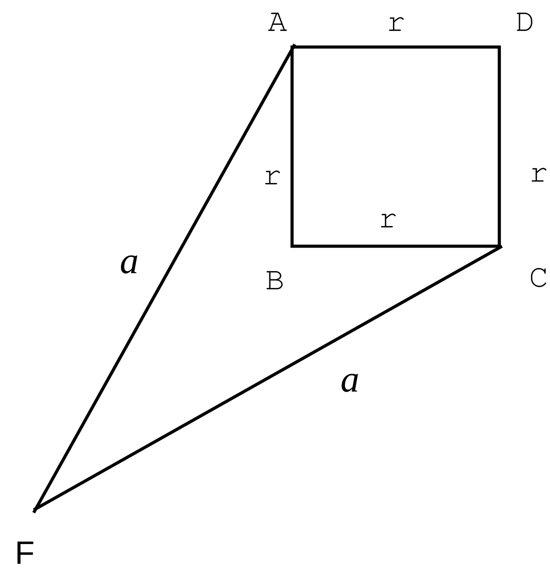

Figure 11: *The Peaucellier inversor.*

By letting $\{D\} = W$ instead of A, the same Z as before, $\lambda = 2$ and taking A as input and G as output we obtain a functional linkage for the function $z \mapsto -z$ in the complex plane. Notice that the condition $s \neq t$ implies that for each realization ϕ the points $\phi(A)$, $\phi(D)$, $\phi(G)$ are pairwise distinct.

Remark 5.2 *Note that zero does not belong to Dom* of the pantograph. To resolve this* *problem we use the translators:*

$$-z = -(z+b) + b = \tau_b(-\tau_b(z))$$

$$\lambda z = \lambda(z+b) - \lambda b = \tau_{-\lambda b}(\lambda \tau'_{-b}(z))$$

$$z/\lambda = (z+b)/\lambda - b/\lambda = \tau_{-b/\lambda}(\tau'_{-b}(z)/\lambda)$$

We call the linkages computing the above functions the modified pantographs *and denote them* $\mathcal{P}'_-, \mathcal{P}'_\lambda, \mathcal{P}'_{1/\lambda}$ *respectively.*

The following is a key lemma which shows that domains of pantographs can be made arbitrarily large, this lemma will be used to prove Theorem on expansion of domain of functional linkages (Theorem 7.2):

Lemma 5.3 *Fix* $\lambda > 1$ *and let* $r > 0$*. Then we can choose* $b \in \mathbb{R}$ *and edge-lengths for the translators and for the pantographs* $\mathcal{P}_\lambda, \mathcal{P}_{1/\lambda}$ *so that*

$$B_r(0) \subset Dom^*(\mathcal{P}'_\lambda), \quad B_r(0) \subset Dom^*(\mathcal{P}'_{1/\lambda})$$

(4) The most famous functional linkage is the Peaucellier inversor (see [HC-V, page 273] and [CR, page 156]) depicted on Figure 11 (with $a^2 - r^2 = t^2$).

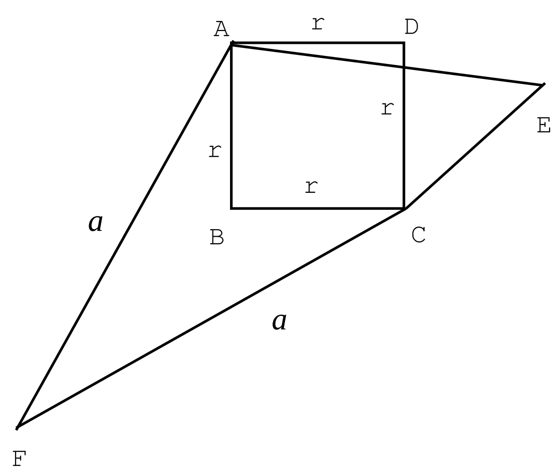

Figure 12: *The modified Peaucellier inversor* $\mathcal{J}_t$*: the square* $ABCD$ *is rigidified and* $\ell[AE] - \ell[EC] = 2\epsilon > 0$, $\ell[EC] > r$.

The vertex F is the only *fixed vertex* of the inversor, $Z := (0)$. According to the 19-th century work on linkages, the Peaucellier inversor is supposed to be functional for the inversion $J_t(z) = t^2/\bar{z}$ with the center at zero and radius t.

Unfortunately this is not true for our definition of functional linkage because of the degenerate realizations ϕ, ψ with $\phi(B) = \phi(D)$ and $\psi(A) = \psi(C)$. Note that there is a 3-torus of degenerate realizations with $\phi(B) = \phi(D)$, so even the dimension of $C(\mathcal{L}, Z)$ is not correct for a functional linkage with $n = m = 1$.

Many of the degenerate realizations can be eliminated by rigidifying the square $ABCD$, but there remains $\mathbb{S}^1 \times \mathbb{S}^1$ of degenerate realizations with $\psi(A) = \psi(C)$ for which $\psi(B)$ and $\psi(D)$ are not in general related by inversion. We eliminate these by attaching a "hook"[5] to $\{A, C\}$ as on the Figure 12.

[5]Notice that by attaching this hook we have created an extra symmetry on the moduli space: the transformation which fixes images of all vertices except $\phi(E)$ and reflects $\phi(E)$ with respect to the line $(\phi(A)\phi(C))$.

Lemma 5.4 *The modified Peaucellier inversor (with B as input and D as output) is functional for* $F(z) = t^2/\bar{z}$.

Remark 5.5 *Note that the origin does not belong to the domain of this linkage.*

(5) *The multiplier.* Guided by the identity

$$\frac{1}{\bar{z}-0.5} - \frac{1}{\bar{z}+0.5} = \frac{1}{\bar{z}^2 - 0.25}$$

we compose the linkages for *translation, inversion, addition* to obtain a functional linkage for germ of the function $F(z) = z^2$ at the origin. Then we use the identity

$$2zw = (z+w)^2 - z^2 - w^2$$

and combine linkages for squaring, addition and the pantograph to construct a functional linkage for complex multiplication.

(6) We obtain the *Peaucellier straight-line motion linkage* $\mathcal{S}$ (Figure 13) as follows:

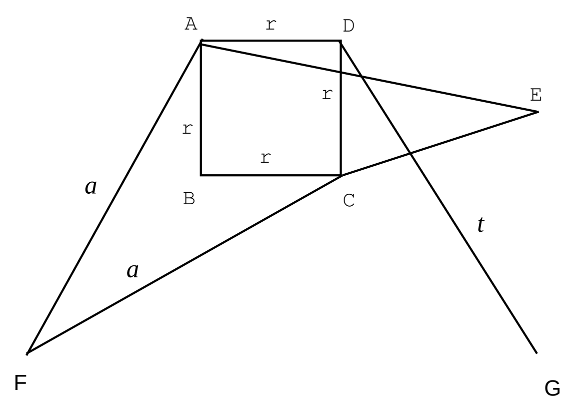

Figure 13: *The Peaucellier straight-line motion linkage:* $t^2 = a^2 - r^2$. *B is the input vertex, the image of B under all realizations lies on a segment of the real axis which contains the open interval* $(-\frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2}t) \subset Dom^*$.

Add the edge $[GD]$ to the with the rigidified inversor $\mathcal{J}_t$. The vertices F, G are the *fixed vertices* of the resulting linkage $\mathcal{S}$. The images of B, D are: $\phi(F) = -\phi(G) = \pm\sqrt{-1}t/2$. Take the vertex B as both the input vertex and the output vertex.

Remark 5.6 *This choice is somewhat strange from the classical point of view since the linkage* $\mathcal{S}$ *was invented to transform circular motion of the vertex D to a periodic linear motion of the vertex B (from this point of view D is the input and B is the output). However for us the input-projection is always onto a domain in the Euclidean space, which is satisfied by B as the input-vertex and* **is not satisfied** *if we take D as the input. The point is that we do not use the linkage* $\mathcal{S}$ *to transform circular to linear motion but to restrict motion of the input-vertex B to the real axis. We obtain a "functional linkage" for the inclusion of the real line into the complex plane (we leave to the reader the necessary modification of Definition 3.7 for this new type of functional linkage).*

The point $\phi(D)$ is now restricted to the circle with the center at $\phi(G)$ and radius $t = \ell[GD] = \ell[FG]$. The input $\phi(B)$ is obtained from $\phi(D)$ by inversion with the center at $\phi(F)$ and radius t. Accordingly the input $\phi(B)$ moves along a segment in the real axis.

We use the following restrictions on the side-lengths of the linkage:

$$0 < 2\epsilon = \ell[AE] - \ell[CE] \ ,$$

$$\ell[CE] > 2r, \quad a > r > \epsilon, \quad 17r > 15a$$

Under these conditions the linkage $\mathcal{S}$ is a real functional linkage for the identity inclusion $id : \mathbb{R} \to \mathbb{C}$ and the input map $p : \mathcal{M}(\mathcal{S}) \to \mathbb{R}^2$ has the following property:

$Dom^*(\mathcal{S})$ contains the open interval $(-\frac{\sqrt{3}}{2}t, \frac{\sqrt{3}}{2}t)$.

The straight-line motion linkage is used for constructing *real* functional linkages from the *complex* ones.

6 Fixing fixed vertices

In this section we explain how to relate the relative configuration spaces $C(\mathcal{L}, Z)$ of marked linkages and the moduli spaces $\mathcal{M}(\mathcal{L})$ of based linkages. Let $\mathcal{L} = (L, \ell, W)$ be a marked linkage, $Z = (z_1, ..., z_s) \in \mathbb{C}^s$ and $W = (w_1, ..., w_s)$. Pick any relative realization $\phi \in C(\mathcal{L}, Z)$.

We first let $\mathcal{L}'$ be the disjoint union of $\mathcal{L}$ and the metric graph $\mathcal{I}$ which consists of a single edge e^* of the unit length connecting the vertices v_1, v_2. Choose the isometric embedding $\phi = \phi_{\mathcal{I}} : \mathcal{I} \to \mathbb{C}$ which maps v_1 to 0 and v_2 to $1 \in \mathbb{R}$. We get a map $\phi : W \cup \mathcal{V}(\mathcal{I}) \to \mathbb{C}$. Then for each pair of vertices $a, b \in W \cup \mathcal{V}(\mathcal{I})$ we do the following:

(a) If $\phi(a) = \phi(b)$ for $\phi \in C(\mathcal{L}, Z)$, we identify the vertices a, b.

(b) Otherwise add to $\mathcal{L}'$ the edge $[ab]$ of the length $|\phi(a) - \phi(b)|$.

Let $\tilde{\mathcal{L}}$ be the resulting based linkage (with the distinguished edge $e^* = [v_1 v_2] \subset \mathcal{I}$).

In the case $\mathcal{L}$ is functional with the input map p we have obvious input map $\tilde{p}$ for the linkage $\tilde{\mathcal{L}}$.

Lemma 6.1 *(i) In the case $Z \notin \mathbb{R}^s$ there is a 2-fold analytically trivial*[6] *covering $\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z)$.*

(ii) In the case $Z \in \mathbb{R}^s$ there is an algebraic isomorphism $\tau : \mathcal{M}(\tilde{\mathcal{L}}) \to C(\mathcal{L}, Z)$.

(iii) Suppose that $\mathcal{L}$ is a (possibly closed) functional linkage for a function f, then $\tilde{\mathcal{L}}$ is again a (possibly closed) functional linkage for f. Moreover, $\tilde{p} = p \circ \tau$ and $Dom^*(\mathcal{L}) = Dom^*(\tilde{\mathcal{L}})$. If p is an algebraic covering then $\tilde{p}$ also is.

7 Expansion of Domains of Functional Linkages

Lemma 7.1 *Suppose that $g(x)$ is a homogeneous polynomial of degree d, $\mathcal{L}$ is a functional linkage which defines the germ $(g, 0)$. Then for any $r > 0$ we can modify $\mathcal{L}$ so that the new linkage $\tilde{\mathcal{L}}$ is functional for the function g and $Dom^*(\tilde{\mathcal{L}})$ contains the disk $B_r(0)$.*

Sketch of the proof: By the assumption $Dom^*(\mathcal{L})$ contains a disk $B_\epsilon(0)$ centered at the origin, we can assume $\epsilon < r$. Choose positive $\lambda < \epsilon/r < 1$. Let $\mu := \lambda^{-d} > 1$. We use the formula

$$g(y) = \lambda^{-d} g(\lambda y) = \mu g(\lambda y)$$

[6] nonalgebraic

to construct a functional linkage $\tilde{\mathcal{L}}$ for the function g as a composition of the following linkages:

- $\mathcal{P}'_\lambda$ (the modified pantograph for the multiplication by λ),
- the linkage $\mathcal{L}$,
- $\mathcal{P}'_\mu$ (the modified pantograph for the multiplication by μ).

Lemma 5.3 is used to ensure that $Dom^*(\tilde{\mathcal{L}})$ contains the disk $B_r(0)$. □

As a corollary we get the following Theorem:

Theorem 7.2 *(Theorem on expansion of domain.) Suppose that $f : \mathbf{k}^m \to \mathbf{k}^n$ be a polynomial morphism, $\mathcal{L}$ is a functional linkage which defines the germ $(f, 0)$. Then for any $r > 0$ we can modify $\mathcal{L}$ so that the new linkage $\tilde{\mathcal{L}}$ is functional for the morphism f and $Dom^*(\tilde{\mathcal{L}})$ contains the disk $B_r(0)$.*

Proof: We consider the case when $n = 1$. Write $f(x)$ as

$$f(x) = \sum_{j \le d} f_j(x)$$

where each f_j is a homogeneous polynomial of degree j. Let $g(y) := y_1 + \dots + y_d$. Hence we can represent f as a composition of homogeneous polynomials $f_j, j \le d$, and g. Now the assertion follows from the previous lemma. □

8 Realization of polynomial morphisms by functional linkages

In this section we sketch a proof of Theorem A. We consider the case of polynomials

$$f : \mathbb{C}^m \to \mathbb{C}, \quad f(x) = a_0 + \sum_j a_j g_j(x)$$

where $g_j = x_1^{\alpha_1} \dots x_m^{\alpha_m}$ are monomials of positive degrees and $a_j \in \mathbb{C}$ are constants ($j = 0, 1, \dots, N$). Let $y = (y_0, \dots, y_N)$. Consider the function

$$\hat{f}(x, y) = y_0 + \sum_j y_j g_j(x)$$

This function is obtained via composition of the multiplication and addition operations. Hence we use the elementary linkages for the addition and multiplication we get a complex functional linkage $\hat{\mathcal{L}}$ for the germ $(\hat{f}, 0)$. Then we use Theorem 7.2: for each given $\rho > 0$ we can modify $\hat{\mathcal{L}}$ to $\tilde{\mathcal{L}}$ so that $\tilde{\mathcal{L}}$ is functional for the pair $(\hat{f}, B_\rho(0))$, $0 \in \mathbb{C}^{m+N}$. We use ρ so large that $B_\rho(0)$ contains the disk

$$\{(x, y) : x \in B_r(\mathcal{O}), y_j = a_j, j = 0, \dots, N\}$$

We represent f as a composition of the function $\hat{f}$ and the constant function

$$a : (y_0, \dots, y_N) \mapsto (a_0, \dots, a_N)$$

The constant function is defined by a functional linkage as follows:

$\mathcal{A}$ is the graph which consists of the set of vertices $[In(\mathcal{A}) = (P_1, ..., P_m)] \cup [Out(\mathcal{A}) = (Q_1, ..., Q_N)]$, has no edges, $W = Out(\mathcal{A})$ and $Z = (a_0, ..., a_N)$.

Composition of the linkages $\tilde{\mathcal{L}}$ and $\mathcal{A}$ gives us a functional linkage for the pair $(f, B_r(\mathcal{O}))$. This proves Theorem A in the complex case when $n = 1$, to prove it for general n we use the fiber sum of linkages. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial function. We extend it to a morphism $f^c : \mathbb{C}^m \to \mathbb{C}^n$ and construct a $\mathbb{C}$-functional linkage $\mathcal{L}$ for f^c. We next alter $\mathcal{L}$ via fiber sums with the Peaucellier straight-line motion linkage $\mathcal{S}$. Namely, take m isomorphic copies $\mathcal{S}_j$ of the linkage $\mathcal{S}$. Then identify each input vertex[7] P_j of $\mathcal{L}$ with the input vertex B_j of $\mathcal{S}_j$. For all $1 \le i < j \le m$ identify F_j and F_i, G_j and G_i. The new linkage $\mathcal{L}'$ has the property:

For each input vertex P_j of $L \subset \mathcal{L}'$ and for all realizations ϕ of $\mathcal{L}'$, the image $\phi(P_j)$ belongs to the real axis $\mathbb{R} \subset \mathbb{C}$. Moreover, suppose that we are given a point $\mathcal{O} = (x_1^0, ..., x_m^0) \in \mathbb{R}^m$, choose the number t (in the definition of $\mathcal{S}$) to be sufficiently large[8], then $\mathcal{L}'$ is a real-functional linkage for the polynomial f and $Dom^*(\mathcal{L}')$ contains the point $\mathcal{O}$. This proves Theorem A for the relative configuration spaces. The assertion about the moduli space follows from the relative case via Lemma 6.1. □

To derive Theorem B from Theorem A we argue as in the Introduction.

9 The Moduli Space of a Planar Arrangement

Let $\mathcal{A}$ be an *arrangement*, i.e. a bipartite graph with parts $\mathcal{P}$ and $\mathcal{L}$. We say that a "point" $p \in \mathcal{P}$ is incident to a "line" $l \in \mathcal{L}$ if p and l are connected by an edge. A projective realization ϕ of $\mathcal{A}$ is a map

$$\phi : \mathcal{P} \cup \mathcal{L} \to \mathbb{P}^2 \cup (\mathbb{P}^2)^\vee, \quad \phi(\mathcal{P}) \subset \mathbb{P}^2, \quad \phi(\mathcal{L}) \subset (\mathbb{P}^2)^\vee$$

such that if p and l are incident then $\phi(p) \in \phi(l)$. We will also use the term *projective arrangements* for projective realizations. When we draw a figure of an arrangement we draw *points* of A as solid points and *lines* as lines.

Definition 9.1 *An arrangement is called* **admissible** *if the bipartite graph has no isolated vertices.*

We let $R(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ denote the set of (complex) projective realizations of $\mathcal{A}$. We have

Lemma 9.2 *$R(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ is the set of complex points of a projective scheme $R(\mathcal{A})$ defined over $\mathbb{Z}$.*

Proof: Let

$$X := \prod_{\mathcal{P}} \mathbb{P}^2 \times \prod_{\mathcal{L}} (\mathbb{P}^2)^\vee$$

and let $I \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ be the incidence relation. Then I is the projective scheme associated to the inverse image of zero for the canonical pairing $\mathbb{A}^3 \times (\mathbb{A}^3)^\vee \to \mathbb{A}$. Then $R(\mathcal{A}) \subset X$ is obtained by imposing the equations defining I for each incident pair (P, L) of vertices of $\mathcal{A}$.
□

We now want to pass to the quotient of $R(\mathcal{A})$ by PGL_3. We do this by restricting to realizations in a "general position" and then taking a cross-section. To make it precise we first define based arrangements.

[7] Here and below we use the symbol X_j to denote the vertex in $\mathcal{S}_j$ corresponding to the vertex X of $\mathcal{S}$.

[8] E.g. larger than $\max_j(|x_j^0|\frac{2}{\sqrt{3}})$.

Definition 9.3 *The* **standard triangle** *is the arrangement T consisting of 6 point-vertices and 6 line-vertices that corresponds to a triangle with its medians, see Figure 14.*

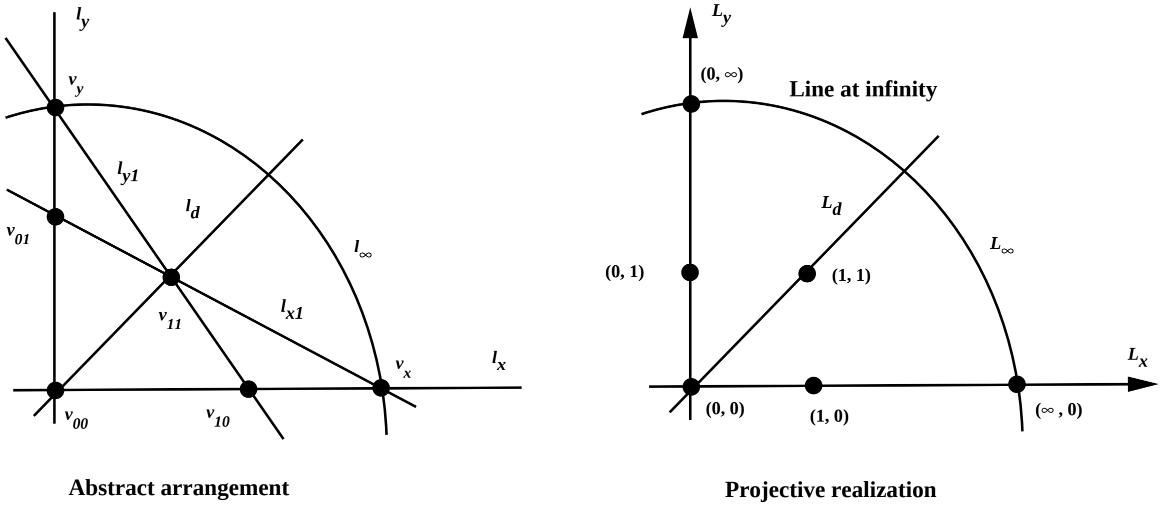

Figure 14: *The standard triangle T and its standard realization.*

Definition 9.4 *The* **standard realization** ϕ_T *of the standard triangle T is determined by:*

$$\phi_T(v_{00}) = (0,0), \phi_T(v_x) = (\infty,0), \phi_T(v_y) = (0,\infty), \phi_T(v_{11}) = (1,1)$$

Here $(0,0), (\infty,0), (0,\infty), (1,1)$ are points in the affine plane $\mathbb{A}^2 \subset \mathbb{P}^2$ which have the homogeneous coordinates: $(0:0:1), (1:0:0), (0:1:0), (1:1:1)$ respectively.

We say that an arrangement $\mathcal{A}$ is *based* if it comes equipped with an embedding $i : T \to \mathcal{A}$. Let $(\mathcal{A}, i)$ be a based arrangement. We say that a projective realization ϕ of $\mathcal{A}$ is *based* if $\phi \circ i = \phi_T$. Let $BR(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ be the subset of $R(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ consisting of based realizations.

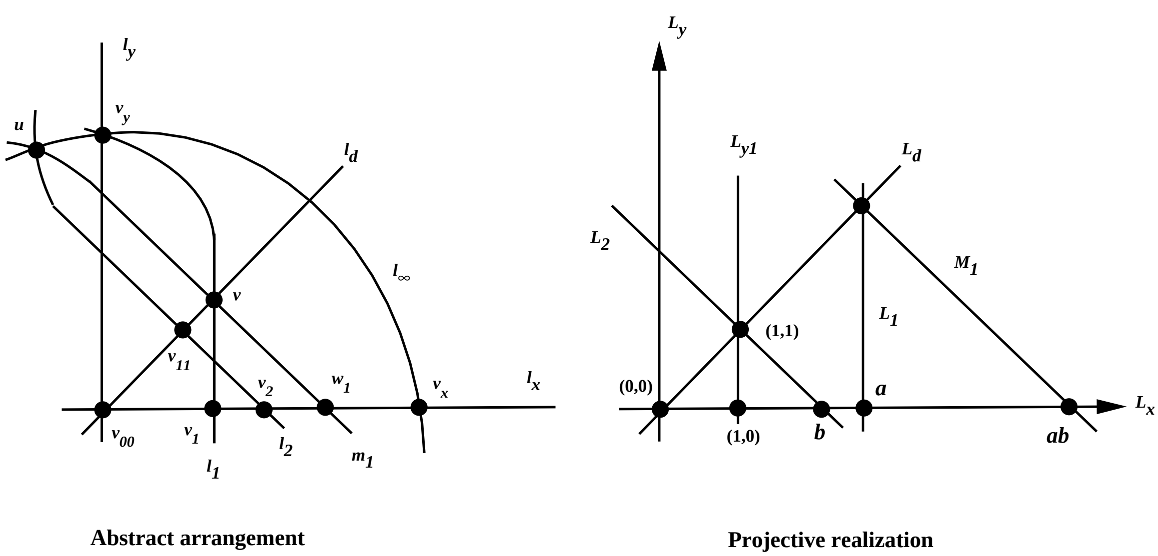

Figure 15: *Arrangement $\mathcal{A}_M$ for the multiplication.*

Lemma 9.5 *$BR(\mathcal{A}, \mathbb{P}^2(\mathbb{C}))$ is the set of complex points of a projective scheme over $\mathbb{Z}$ which is a scheme-theoretic quotient of $R(\mathcal{A})$ by the action of PGL_3.*

Proof: Let $U \subset R(\mathcal{A})$ be Zariski open subscheme such that $\phi(v_{00}), \phi(v_x), \phi(v_y), \phi(v_{11})$ are in general position. Clearly

$$BR(\mathcal{A}, \mathbb{P}^2(\mathbb{C})) = U(\mathbb{C})/PGL_3(\mathbb{C})$$

But in fact U is the set of stable points for an appropriate projective embedding $R(A) \hookrightarrow \mathbb{P}^N$ (see [KM6, §8.5]). There are no semistable points which are not stable in this case, whence U/PGL_3 is projective. □

Suppose that $(\mathcal{A}, i)$ is a based arrangement and $\mu \subset \mathcal{P}$ is a collection of vertices incident to l_x. We get a *marked* based arrangement $\mathcal{A}_\mu$ (to simplify the notation we will sometimes drop the subscript μ for marked arrangements). We define the Zariski open subscheme $BR_0(\mathcal{A}_\mu)$ of *finite* (relative to μ) realizations by requiring that for all $P_j \in \mu$ and $\psi \in BR_0(\mathcal{A}_\mu)$, $\psi(P_j)$ is not in the line at infinity. Now we can define functional arrangements.

Definition 9.6 *A* **functional arrangement** *is a based arrangement $(\mathcal{A}, i)$ with two subsets of marked point-vertices $\mu = (P_1, ..., P_m)$ and $\nu = (Q_1, ..., Q_n)$ such that all the marked vertices are incident to the line-vertex $l_x \in i(T)$ (which corresponds to the x-axis) and such that the following two axioms are satisfied:*

(1) $BR_0(\mathcal{A}_\mu) \subset BR_0(\mathcal{A}_\nu)$.

(2) The projection $p : BR_0(\mathcal{A}_\mu) \to \mathbb{A}^m$ given by $p(\phi) = (\phi(P_1), ..., \phi(P_m))$ is an isomorphism of schemes over $\mathbb{Z}$.

Each functional arrangement determines a morphism $f : \mathbb{A}^m \to \mathbb{A}^n$ (which is defined over $\mathbb{Z}$) by the formula:

$$f(x) = q \circ p^{-1}(x)$$

where $q(\phi) = (\phi(Q_1), ..., \phi(Q_n))$.

10 Realization of Affine Schemes as Moduli Spaces of Arrangements

We can compose functional arrangements by gluing[9] outputs and inputs, hence (as in the case of linkages) in order to prove Theorem D it suffices to produce functional arrangements for addition and multiplication as well as for the constant functions $f_\pm(z) = \pm 1$. These arrangements come from the classical projective geometry [H] and [St], see Figures 15, 16. The scheme-theoretic proofs are to be found in [KM6, Theorem 9.1]. For instance, for any projective realization $\phi \in BR_0(\mathcal{A}_A)$ of the arrangement $\mathcal{A}_A$ (which is functional for addition), the images of v_1, v_2, w_1 are related by the formula:

$$\phi(v_1) + \phi(v_2) = \phi(w_1)$$

To obtain a functional arrangement $\mathcal{A}_h$ for the function $h(x, y, z) = (x + y)z$ we take the fiber sum of $\mathcal{A}_A$ and $\mathcal{A}_M$ where we identify the output $w_1 \in \mathcal{A}_A$ and the input $v_1 \in \mathcal{A}_M$. To obtain a functional arrangement $\mathcal{A}_g$ for the function $g(x, y) = (x + y)x$ we take the arrangement $\mathcal{A}_h$ and identify the input $v_1 \in \mathcal{A}_M$ with the input $v_1 \in \mathcal{A}_A$. For details of the proof of Theorem D see [KM6, Section 9].

Let S be a closed subscheme (over $\mathbb{Z}$) in $\mathbb{A}^m$ defined by the system of equations:

$$\begin{cases} f_1(x_1, ..., x_m) = 0 \\ f_2(x_1, ..., x_m) = 0 \\ \vdots \\ f_n(x_1, ..., x_m) = 0 \end{cases}$$

[9]Abstractly speaking such gluing is a fiber sum of arrangements.

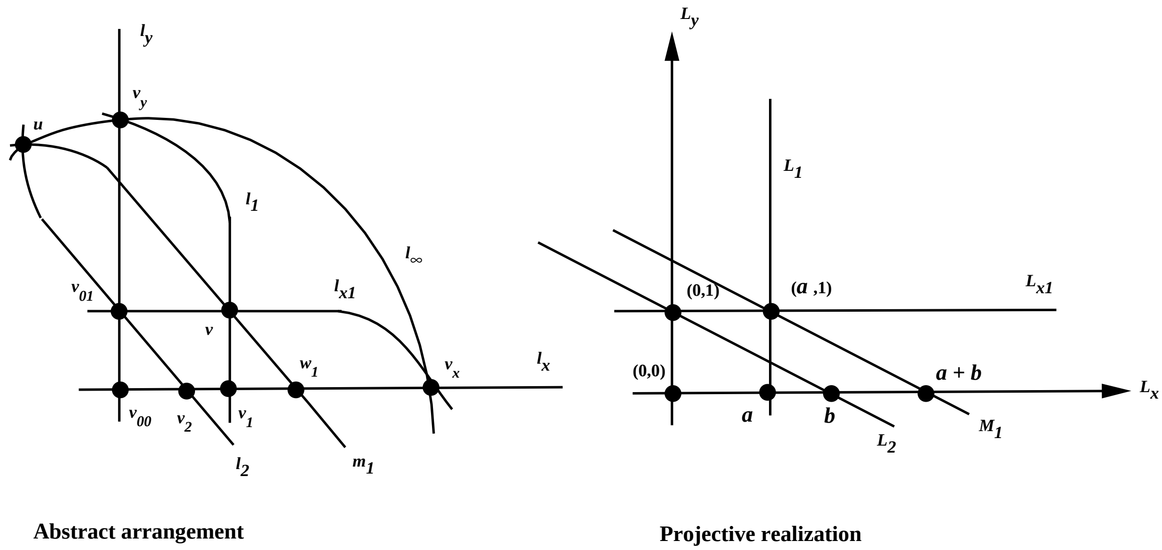

Figure 16: *Arrangement $\mathcal{A}_A$ for addition.*

By Theorem D we have a functional arrangement $\mathcal{A}_f$ for the vector-function f. Consider the linkage $\mathcal{A}_0$ obtained by gluing the output vertices of $\mathcal{A}$ to the vertex v_{00}. This amounts to restricting realizations $\phi \in BR_0(\mathcal{A}_f)$ to those for which $q(\phi) = 0$ or, equivalently, $f(p(\phi)) = 0$. Thus the input-projection $p : BR_0(\mathcal{A}_0) \to \mathbb{A}^n$ induces an isomorphism $BR_0(\mathcal{A}_0) \to S$ and we obtain Theorem E (the scheme-theoretic version of Mnev's Theorem). For instance, to obtain an arrangement such that $BR_0(\mathcal{A}_0) \cong \{x^2 = 0\} \subset \mathbb{A}^1$ we take the arrangement $\mathcal{A}_M$ for the multiplication, identify the vertices v_1 and v_2 (this gives us an arrangement for the function $f(x) = x^2$) and then glue the vertex w_1 to v_{00}. This produces the required arrangement $\mathcal{A}_0$.

To prove Theorem F we combine Theorems E and 2.1. We apply Theorem 2.2 to deduce Corollary G.

For our application to Serre's problem we will need the following. Suppose that X is an affine scheme of finite type over $\mathbb{Z}$ which has an integer point $x \in X$. Assume X is realized in $\mathbb{A}^m$ so that $x = O$ is the origin. Thus X is defined by a system of polynomial equations with integer coefficients without constant terms. Let $\mathcal{A}$ be a based marked arrangement so that $BR_0(\mathcal{A}) \cong X$. Let $\phi_0 \in BR_0(\mathcal{A})$ be the realization corresponding to $O \in X$. Then we have

Lemma 10.1 *We may choose $\mathcal{A}$ such that $\phi_0(\mathcal{A}) = \phi_0 \circ i(T)$.*

Proof: To prove this lemma we will need a slight modification of the previous construction.

We first write down polynomial equations defining $X \subset \mathbb{A}^m$ so that the formulae involve only addition and multiplication and no multiplicative and additive constants. Namely, suppose that we have a system of polynomial equations

$$\begin{cases} f_1(x_1, ..., x_m) = 0 \\ f_2(x_1, ..., x_m) = 0 \\ \vdots \\ f_n(x_1, ..., x_m) = 0 \end{cases}$$

Represent each polynomial f_j as the difference $f_j^+ - f_j^-$ where $f_j^\pm$ have only positive coefficients. Then our system of equations is equivalent to

$$\begin{cases} f_1^+(x) = f_1^-(x) \\ f_2^+(x) = f_2^-(x) \\ \vdots \\ f_n^+(x) = f_n^-(x) \end{cases}$$

We can write down each polynomial function $f_j^\pm$ as the sum of monomials without multiplicative constants:

$$\sum_\alpha n_\alpha x^\alpha = \sum_\alpha (\underbrace{x^\alpha + ... + x^\alpha}_{n_\alpha \text{ times}})$$

where α is a multi-index $(k_1, ..., k_m)$, $x^\alpha = x^{k_1} ... x^{k_m}$. For instance the equation $2x^2 - y = 0$ will be rewritten as $x^2 + x^2 = y$.

Let $F^+ := (f_1^+, ..., f_n^+), F^- := (f_1^-, ..., f_n^-)$. Take functional arrangements $\mathcal{A}_{F^+}, \mathcal{A}_{F^-}$ for these vector-functions. They have the output vertices $Q_1^+, ..., Q_n^+$, $Q_1^-, ..., Q_n^-$ corresponding to the functions $f_j^\pm$. Let $\mathcal{A}$ be the arrangement obtained from $\mathcal{A}_{F^+}, \mathcal{A}_{F^-}$ by identifying Q_j^+ and Q_j^- for each $j = 1, ..., n$.

Then $\mathcal{A}$ is a fiber sum of the basic arrangements for the addition and multiplication. We notice that if we specialize the realizations of input-vertices of the two basic arrangements (for the addition and multiplication) to zero then the conclusion of Lemma holds for these arrangements. Hence the lemma holds for the fiber sums of these arrangements as well. $\square$

11 Coxeter, Shephard and Artin groups

Let Λ be a finite graph where two vertices are connected by at most one edge, there are no loops (i.e. no vertex is connected by an edge to itself) and each edge e is assigned an integer $\epsilon(e) \geq 2$. We call Λ a *labelled* graph, let $\mathcal{V}(\Lambda)$ and $\mathcal{E}(\Lambda)$ denote the sets of vertices and edges of Λ. When drawing Λ we will omit labels 2 from the edges (since in our examples most of the labels are 2). Given Λ we construct two finitely-presented groups corresponding to it. The first group G_Λ^c is called the *Coxeter group* with the *Coxeter graph* Λ, the second is the *Artin group* G_Λ^a. The sets of generators for the both groups are $\{g_v, v \in \mathcal{V}(\Lambda)\}$. Relations in G_Λ^c are:

$$g_v^2 = 1, v \in \mathcal{V}(\Lambda), \ (g_v g_w)^{\epsilon(e)} = \mathbf{1}, \quad \text{over all edges} \quad e = [vw] \in \mathcal{E}(\Lambda)$$

Relations in G_Λ^a are:

$$\underbrace{g_v g_w g_v g_w ...}_{\epsilon \text{ terms}} = \underbrace{g_w g_v g_w g_v ...}_{\epsilon \text{ terms}}, \quad \epsilon = \epsilon(e), \ \text{over all edges} \quad e = [vw] \in \mathcal{E}(\Lambda)$$

For instance, if we have an edge $[vw]$ with the label 4, then the Artin relation is

$$g_v g_w g_v g_w = g_w g_v g_w g_v$$

Note that there is an obvious epimorphism $G_\Lambda^a \longrightarrow G_\Lambda^c$. We call the groups G_Λ^c and G_Λ^a *associated* with each other. The Artin groups appear as generalizations of the Artin braid group. Each Coxeter group G_Λ^c admits a canonical discrete faithful linear representation

$$h : G_\Lambda^c \longrightarrow GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$$

where n is the number of vertices in Λ. Suppose that the Coxeter group G^c_Λ is finite, then remove from $\mathbb{C}^n$ the collection of fixed points of elements of $h(G^c_\Lambda - \{\mathbf{1}\})$ and denote the resulting complement X_Λ. The group G^c_Λ acts freely on X_Λ and the quotient X_Λ/G^c_Λ is a smooth complex quasi-projective variety with the fundamental group G^a_Λ, see [B] for details. Thus the Artin group associated to a finite Coxeter group is the fundamental group of a smooth complex quasi-projective variety.

The construction of Coxeter and Artin groups can be generalized as follows. Suppose that not only edges of Λ, but also its vertices v_j have labels $\delta_j = \delta(v_j) \in \{0, 2, 3, ...\}$. Then take the presentation of the Artin group G^a_Λ and add the relations:

$$g_v^{\delta(v)} = \mathbf{1}, \quad v \in \mathcal{V}(\Lambda)$$

If $\delta(v) = 2$ for all vertices v then we get the Coxeter group, in general the resulting group is called the *Shephard group*, they were introduced by Shephard in [Sh]. Again there is a canonical epimorphism $G^a_\Lambda \to G^s_\Lambda$.

12 Groups Corresponding to Abstract Arrangements

Suppose that $\mathcal{A}$ is a based arrangement. We start by identifying the point-vertex v_{00} with the line-vertex l_∞, the point-vertex v_x with the line-vertex l_y and the point-vertex v_y with the line-vertex l_x in the standard triangle T. We also introduce the new edges

$$[v_{10}v_{00}], \quad [v_{01}v_{00}], \quad [v_{11}v_{00}]$$

(Where $v_{10}, v_{00}, v_{11}, v_{01}$ are again *points* in the standard triangle T.) We will use the notation Λ for the resulting graph. Put the following labels on the edges of Λ:

1) We assign the label 4 to the edges $[v_{10}v_{00}], [v_{01}v_{00}]$ and all the edges which contain v_{11} as a vertex (with the exception of $[v_{11}v_{00}]$). We put the label 6 on the edge $[v_{11}v_{00}]$.

2) We assign the label 2 to the rest of the edges.

Now we have labelled graphs and we use the procedure from the Section 11 to construct:

(a) The *Artin group* $G^a_{\mathcal{A}} := G^a_\Lambda$.

(b) We assign the label 3 to the vertex v_{11} and labels 2 to the rest of the vertices. Then we get the *Shephard* group $G^s_{\mathcal{A}} := G^s_\Lambda$.

We will denote the generators of the above groups g_v, g_l, where v, l are elements of $\mathcal{A}$ (corresponding to the vertices of Λ).

13 Representations Associated with Anisotropic Projective Arrangements

Let q be the quadratic form $x_1^2 + x_2^2 + x_3^2$ and $PO(3)$ be the projectivized group of isometries of q. From now on we work over $\mathbb{Q}$ (rather than $\mathbb{Z}$). Let Z be the projectivized null quadric of q and $\mathbb{P}^2_0 = \mathbb{P}^2 - Z$. We let $(\mathbb{P}^2_0)^\vee$ be the image of $\mathbb{P}^2_0$ under the polarity defined by q. A projective arrangement ψ will be said to be *anisotropic* if $\psi(P) \in \mathbb{P}^2_0, \psi(L) \in (\mathbb{P}^2_0)^\vee$, for all $P \in \mathcal{P}, L \in \mathcal{L}$. The anisotropic condition defines Zariski open subschemes of the arrangement varieties to be denoted $R(\mathcal{A}, \mathbb{P}^2_0), BR(\mathcal{A}, \mathbb{P}^2_0)$ and $BR_0(\mathcal{A}, \mathbb{P}^2_0)$ respectively.

Now a point P in $\mathbb{P}^2_0$ determines the Cartan involution σ_P in $PO(3)$ around this point or the rotation θ_P of order 3 having this point as neutral fixed point (i.e. a point where the differential of rotation has the determinant 1). There are two such rotations of order 3, we choose one of them. Since ψ is *based*, $\psi(v_{11}) = (1, 1)$ for all ψ, hence the choice of rotation is harmless (see [KM6, §12.1]). Similarly a line $L \in (\mathbb{P}_0^2)^\vee$ uniquely determines the reflection σ_L which keeps L pointwise fixed. Finally one can encode the incidence relation between points and lines in $\mathbb{P}^2$ using algebra: two involutions generate the subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$ in $PO(3)$ iff the neutral fixed point of one belongs to the fixed line of another, rotations σ, θ of orders 2 and 3 anticommute (i.e. $\sigma\theta\sigma\theta = 1$) iff the neutral fixed point of the rotation θ belongs to the fixed line of the involution σ, etc. Let $G^s_{\mathcal{A}}$ denote the Shephard group corresponding to the arrangement $\mathcal{A}$. We get the *algebraization* morphism

$$alg: \text{ based anisotropic arrangements } \longrightarrow \text{ representations of } G^s_{\mathcal{A}}$$

$$alg: \psi \mapsto \rho, \rho(g_v) = \sigma_{\psi(v)}, v \in \mathcal{V}(\Lambda) - \{v_{11}\}, \rho(g_{v_{11}}) = \theta_{\psi(v_{11})},$$
$$\rho \in \mathrm{Hom}(G^s_{\mathcal{A}}, PO(3)), \psi \in BR(\mathcal{A})$$

If Γ is a finitely-generated group and G is an algebraic Lie group then

$$X(\Gamma, G) := \mathrm{Hom}(\Gamma, G)//G$$

will denote the character variety of representations $\Gamma \to G$.

Theorem 13.1 *The mapping* $alg: BR(\mathcal{A}, \mathbb{P}_0^2) \to X(G^s_{\mathcal{A}}, PO(3))$ *is an isomorphism onto a Zariski open and closed subvariety to be denoted* $\mathrm{Hom}_f^+(G^s_{\mathcal{A}}, PO(3))//PO(3)$.

Remark 13.2 *In [KM6] we give an explicit description of* $\mathrm{Hom}_f^+(G^s_{\mathcal{A}}, PO(3))$.

The mapping alg has the following important property: Let X be an affine scheme defined over $\mathbb{Z}$ and $O \in X$ be an integer point. Choose an embedding (defined over $\mathbb{Z}$) $X \hookrightarrow \mathbb{A}^m$ into affine space such that O goes to the origin. Let $\mathcal{A}$ be an arrangement corresponding to $X \subset \mathbb{A}^m$ as in Lemma 10.1 and $\phi_0 \in BR_0(A)$ correspond to the origin under the isomorphism $\tau: BR_0(\mathcal{A}) \to X$ given by Theorem E.

Lemma 13.3 *The image of* $G^s_{\mathcal{A}}$ *under* $\rho_0 = alg(\phi_0)$ *is finite.*

Proof: It follows from Lemma 10.1 that $\phi_0(\mathcal{A}) = \phi_0 \circ i(T)$. Then it is straightforward to verify that the group $\rho_0(G^s_{\mathcal{A}}) = \rho_0(G^s_T)$ is isomorphic to the alternating group on four letters. □

It remains to examine the morphism

$$\mu: \mathrm{Hom}_f^+(G^s_{\mathcal{A}}, PO(3))//PO(3) \to X(G^a_{\mathcal{A}}, PO(3))$$

given by pull-back of homomorphisms.

Theorem 13.4 *Suppose that $\mathcal{A}$ is an* **admissible** *based arrangement. Then the morphism μ is an isomorphism onto a union of Zariski connected components.*

Proof: See [KM6, Theorem 12.26]. □

Corollary 13.5 *The character variety* $X(G^a_{\mathcal{A}}, PO(3))$ *inherits all the singularities of the character variety* $X(G^s_{\mathcal{A}}, PO(3))$ *corresponding to points of* $BR(\mathcal{A}, \mathbb{P}_0^2)$.

14 Examples of Artin Groups That Are Not Fundamental Groups of Smooth Complex Algebraic Varieties

Theorem 14.1 *There are infinitely many mutually nonisomorphic Artin (and Shephard) groups which are not isomorphic to the fundamental groups of smooth connected complex algebraic varieties*[10].

To prove this theorem we apply our version of a theorem of R. Hain [Hai].

Theorem 14.2 *Suppose M is a (not necessarily compact) smooth connected complex algebraic variety, G is a reductive algebraic Lie group defined over $\mathbb{R}$ and $\rho : \pi_1(M) \to G$ is a representation with finite image. Then the germ $(\mathrm{Hom}(\pi_1(M), G), \rho)$ is a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4. Suppose further that there is a local cross-section through ρ to the $Ad(G)$-orbits in $\mathrm{Hom}(\pi_1(M), G)$. Then the quotient germ $(\mathrm{Hom}(\pi_1(M), G)//G, [\rho])$ is a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4.*

Proof: See [KM6, Theorem 15.1]. □

Remark 14.3 *The reader will find a discussion of Hain's unpublished work in [KM6, §14]. We give two different proofs of Theorem 14.2: one based on Hain's work and the other based on the results of Morgan [Mo].*

In Theorem 14.2 we use the following definitions:

Definition 14.4 *Let X be a real or complex analytic space $x \in X$ and G a Lie group acting on X. We say that there is a* **local cross-section** *through x to the G-orbits if there is a G-invariant open neighborhood U of x and a closed analytic subspace $S \subset U$ such that the natural map $G \times S \to U$ is an isomorphism of analytic spaces.*

Suppose that we have a collection of polynomials $F = (f_1, ..., f_m)$ of n variables, we assume that all these polynomials have trivial linear parts. The polynomial f_j is said to be *weighted homogeneous* if there is a collection of positive integers (weights) $w_1 > 0, ..., w_n > 0$ and a number $u_j \geq 0$ so that

$$f_j((x_1 t^{w_1}), ..., (x_n t^{w_n})) = t^{u_j} f_j(x_1, ..., x_n)$$

for all t. Let Y denote the scheme given by the system of equations

$$\{f_1 = 0, ..., f_m = 0\}$$

We say that $(Y, 0)$ is a *quasi-homogeneous* if we can choose generators $f_1, ..., f_m$ for its defining ideal such that all the polynomials f_j are weighted homogeneous with the same weights $w_1, ..., w_n$ (we do not require u_j to be equal for distinct $j = 1, ..., m$). We will call the numbers w_i the *weights of generators* and the numbers u_j the *weights of relations.*

To prove Theorem 14.1 we use the (nonreduced) singularities $V_p := \{x^p = 0\}$ where $p \geq 5$ are prime numbers. They correspond to arrangements $\mathcal{A}_p$ so that $BR_0(\mathcal{A}_p) \cong V_p$ (as in Lemma 10.1). Then take the corresponding Shephard and Artin groups $G^s_{\mathcal{A}_p}, G^a_{\mathcal{A}_p}$. It follows that the point $0 \in V_p$ corresponds in the character varieties $X(G^s_{\mathcal{A}_p}, PO(3))$, $X(G^s_{\mathcal{A}_p}, PO(3))$ to (the equivalence classes of) the finite representations ρ^s, ρ^a of the groups $G^s_{\mathcal{A}_p}, G^a_{\mathcal{A}_p}$. The

[10] Which are not necessarily compact.

singularities of these character varieties near ρ^s, ρ^a are analytically isomorphic to V_p. Now Theorem 14.1 follows from Theorem 14.2.

Below is a specific example. Consider the labelled graph on Figure 17 where:

(1) All vertices labelled by the same letter are identified.

(2) The unlabeled edges are to be labelled by 2.

The Artin group $G^a_{\mathcal{A}}$ associated to this graph has the property that the singularity of the character variety $X(G^a_{\mathcal{A}}, PO(3))$ at the equivalence class of the representation $\mu \circ alg(\phi_0)$ is analytically equivalent to $V_5 = \{x^5 = 0\}$. Hence by Theorem 14.2 the group $G^a_{\mathcal{A}}$ is not the fundamental group of a smooth complex algebraic variety.

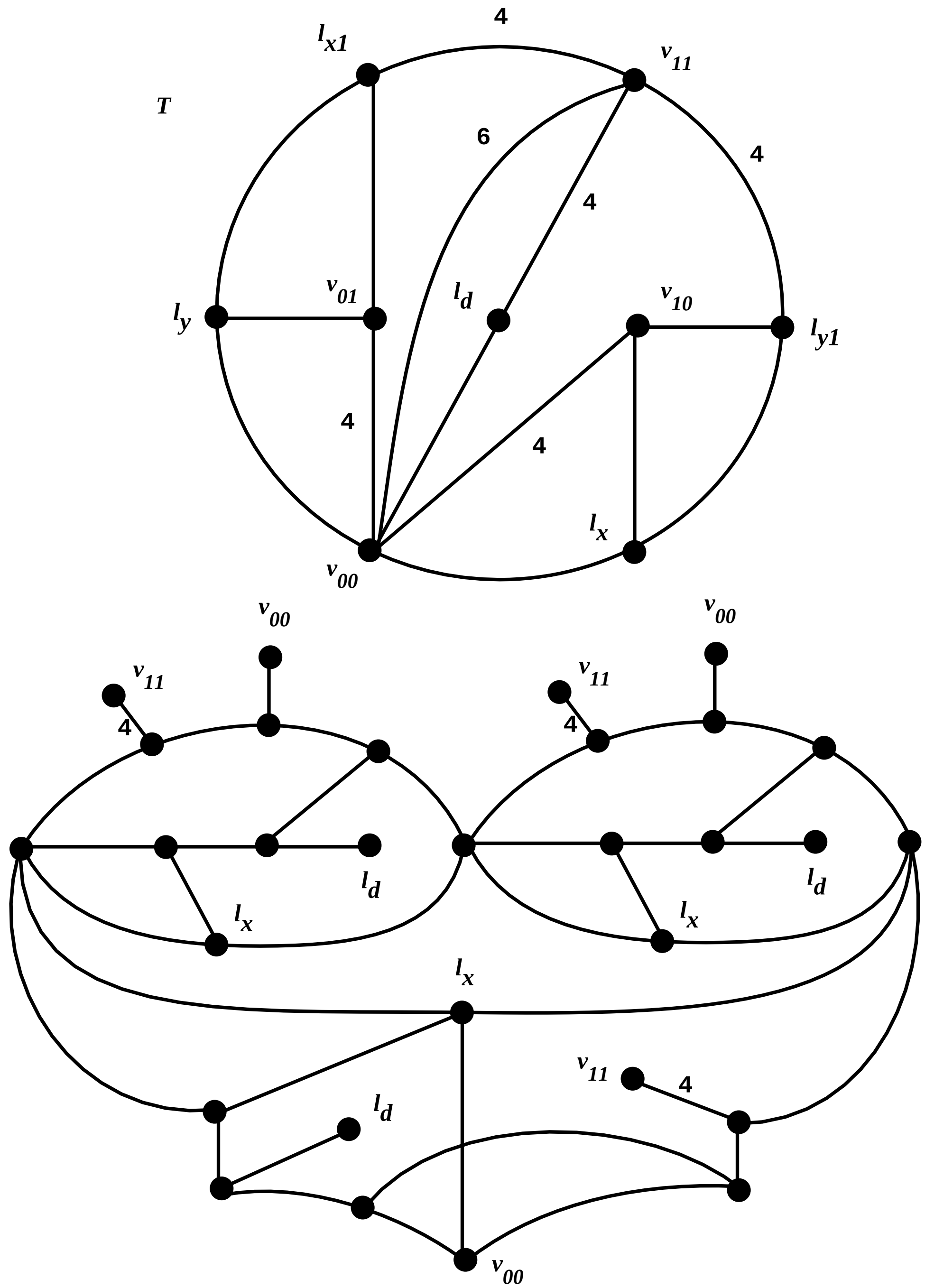

Figure 17: *Labelled graph of an Artin group.*

15 Relation Between the Two Universality Theorems

The goal of this section is to outline a relation between the two universality theorems for realizability of real algebraic sets (Theorems B and F), the details can be found in [KM8]. Consider a based arrangement $\mathcal{A}$. We construct a metric graph L corresponding to $\mathcal{A}$ as follows. As in §12 we identify the point-vertex v_{00} with the line-vertex l_∞, the point-vertex v_x with the line-vertex l_y and the point-vertex v_y with the line-vertex l_x in the standard triangle T. We introduce the new edges

$$[v_{10}v_{00}], \quad [v_{01}v_{00}], \quad [v_{10}v_x], \quad [v_{01}v_y]$$

Let L be the resulting graph. We construct a length-function ℓ on $\mathcal{E}(L)$ as follows:

1) We assign the length $\pi/4$ to the new edges.

2) We assign the length $\pi/2$ to the rest of the edges.

We choose $v_{00}, v_x, v_y, v_{01}, v_{10}$ as the distinguished vertices of the corresponding metric graph L. Let $\mathcal{L}$ denote the marked metric graph L with the distinguished set of vertices as above. Let X be either $\mathbb{S}^2$ or $\mathbb{RP}^2$ with the standard metric d (so that the standard projection $\mathbb{S}^2 \to \mathbb{RP}^2$ is a local isometry). Define the *configuration space* $C(\mathcal{L}, X)$ of realizations of $\mathcal{L}$ in X to be the collection of mappings ψ from the vertex-set $\mathcal{V}(\mathcal{L})$ of $\mathcal{L}$ to X such that

$$d(\psi(v), \psi(w))^2 = (\ell[vw])^2$$

for all vertices v, w of $\mathcal{L}$ connected by an edge.

Remark 15.1 *Notice that if $a, b \in \mathbb{RP}^2$ have distance $\pi/2$ between them then there are two minimal geodesics connecting a to b. This is the reason to define $C(\mathcal{L}, X)$ as the set of maps from $\mathcal{V}(\mathcal{L})$ rather than from $\mathcal{L}$ itself.*

One can easily see that $C(\mathcal{L}, X)$ has a natural structure of a real algebraic set. The subsets

$$\mathcal{M}(\mathcal{L}, \mathbb{RP}^2) := \{\psi \in C(\mathcal{L}, \mathbb{RP}^2) : \psi(v_{00}) = (0,0), \psi(v_x) = (\infty, 0), \psi(v_{10}) = (1,0), \psi(v_{01}) = (0,1)\}$$

$$\mathcal{M}(\mathcal{L}, \mathbb{S}^2) := \{\psi \in C(\mathcal{L}, \mathbb{S}^2) : \psi(v_{00}) = (0,0,1), \psi(v_y) = (0,1,0), \psi(v_x) = (1,0,0), \psi(v_{10}) = (1,0,1), \psi(v_{01}) = (0,1,1)\}$$

form cross-sections to the actions of the groups of isometries $PO(3,\mathbb{R}), O(3,\mathbb{R})$ of X on $C(\mathcal{L}, X)$. We call $\mathcal{M}(\mathcal{L}, X)$, the *moduli spaces* of realizations of $\mathcal{L}$ in X (where $X = \mathbb{S}^2, \mathbb{RP}^2$).

Remark 15.2 *Now it is convenient to use the full group of isometries of $\mathbb{S}^2$ instead of the group of orientation-preserving isometries that we used for planar linkages.*

The next lemma follows from the fact that a point $P \in \mathbb{RP}^2$ is incident to a line $L \in (\mathbb{RP}^2)^\vee$ iff

$$d(P, L^\vee) = \pi/2$$

Lemma 15.3 *The moduli space $\mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$ is algebraically isomorphic to the real algebraic set $BR(\mathcal{A}, \mathbb{RP}^2)$.*

Let $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ be the image of $BR_0(\mathcal{A}_0, \mathbb{RP}^2)$ under the isomorphism given by the previous lemma. Consider the standard 2-fold covering $\mathbb{S}^2 \to \mathbb{RP}^2$. It induces a (locally trivial) analytical covering

$$\alpha : \mathcal{M}(\mathcal{L}, \mathbb{S}^2) \to \mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$$

The group of automorphisms of α is $(\mathbb{Z}_2)^r$, where r is the number of (point) vertices in $[L - \mathcal{P}(T)] \cup \{v_{11}\}$. The generators of this group are indexed by the vertices $v \in [L - \mathcal{P}(T)] \cup \{v_{11}\}$:

$$g_v : \psi(v) \mapsto -\psi(v); \; g_v : \psi(w) \mapsto \psi(w), w \neq v$$

Proposition 15.4 *For each arrangement $\mathcal{A}$ as in Theorem F, the covering α is analytically trivial over $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$.*

Proof: The following fact implies the proposition:

Let v be a vertex of $\mathcal{L}$. Then there is a projective line λ in $\mathbb{P}^2$ (if v is a point-vertex) or in $(\mathbb{P}^2)^\vee$ (if v is a line-vertex) so that $\phi(v) \notin \lambda$ for all $\phi \in BR_0(\mathcal{A}, \mathbb{RP}^2)$.

It is enough to verify the above property for the arrangements $\mathcal{A}_A, \mathcal{A}_M$ for the addition and multiplication which is straightforward. □

Now we identify the moduli space of spherical linkages $\mathcal{M}(\mathcal{L}, \mathbb{S}^2)$ with a moduli space of Euclidean linkages in $\mathbb{R}^3$ as follows:

Add an extra vertex v_0 to the graph $\mathcal{L}$ and connect it to each vertex of $\mathcal{L}$ by edge of the unit length. Modify the other side-lengths as follows:

$$\ell'(e) := \sqrt{2 - 2\cos(\ell(e))}, \quad e \in \mathcal{E}(\mathcal{L})$$

Let $\mathcal{L}'$ be the resulting metric graph with the distinguished set of vertices $[\mathcal{P}(T) - \{v_{11}\}] \cup \{v_0\}$. Define the configuration space

$$C(\mathcal{L}', \mathbb{R}^3) := \{\psi : \mathcal{V}(\mathcal{L}') \to \mathbb{R}^3 : |\psi(v) - \psi(w)|^2 = \ell'[vw]^2\}$$

Again is is clear that

$$\mathcal{M}(\mathcal{L}', \mathbb{R}^3) := \{\psi \in C(\mathcal{L}', \mathbb{R}^3) : \psi(v_0) = (0,0,0),$$

$$\text{and the same normalization on } \mathcal{P}(T) - \{v_{11}\} \text{ as we used for } \mathcal{M}(\mathcal{L}, \mathbb{S}^2)\}$$

is a real-algebraic set which is a cross-section for the action of $Isom(\mathbb{R}^3)$ on $C(\mathcal{L}', \mathbb{R}^3)$. Obviously we have an algebraic isomorphism

$$\mathcal{M}(\mathcal{L}, \mathbb{S}^2) \cong \mathcal{M}(\mathcal{L}', \mathbb{R}^3)$$

of real-algebraic sets. We let $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3)$ be the subset of $\mathcal{M}(\mathcal{L}', \mathbb{R}^3)$ corresponding to $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ under the isomorphism

$$\mathcal{M}(\mathcal{L}, \mathbb{RP}^2) \cong \mathcal{M}(\mathcal{L}, \mathbb{S}^2) \cong \mathcal{M}(\mathcal{L}', \mathbb{R}^3)$$

We obtain

Theorem 15.5 *Let S be a compact real-algebraic set defined over $\mathbb{Z}$. Then there are linkages $\mathcal{L}, \mathcal{L}'$ so that:*

(1) $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$ is entirely isomorphic to S.

(2) $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3)$ is an (analytically) trivial entire rational covering of S.

Both $\mathcal{M}_0(\mathcal{L}, \mathbb{RP}^2)$, $\mathcal{M}_0(\mathcal{L}', \mathbb{R}^3)$ are Zariski open and closed subsets in the moduli spaces $\mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$, $\mathcal{M}(\mathcal{L}', \mathbb{R}^3)$ respectively.

Corollary 15.6 *Suppose that M is a smooth compact manifold. Then there are linkages $\mathcal{L}, \mathcal{L}'$ so that M is diffeomorphic to unions of components in $\mathcal{M}(\mathcal{L}, \mathbb{RP}^2)$, $\mathcal{M}(\mathcal{L}', \mathbb{R}^3)$.*

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