Morse actions of discrete groups on symmetric spaces: Local-to-global principle

Michael Kapovich, Bernhard Leeb, Joan Porti

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Abstract

Our main result is a local-to-global principle for Morse quasigeodesics, maps and actions. As an application of our techniques we show algorithmic recognizability of Morse actions and construct Morse "Schottky subgroups" of higher rank semisimple Lie groups via arguments not based on Tits' ping-pong. Our argument is purely geometric and proceeds by constructing equivariant Morse quasiisometric embeddings of trees into higher rank symmetric spaces.

Contents

1 Introduction

This is a sequel to our paper [KLP5] and mostly consists of the material of section 7 of our earlier paper [KLP1] (the only additional material appears in Theorem 4.8 and the appendix to the paper). We recall that quasigeodesics in Gromov hyperbolic spaces can be recognized locally by looking at sufficiently large finite pieces, see [CDP]. In our earlier papers [KLP4, KLP5, KLP2, KL1, KL2], for higher rank symmetric spaces X (of noncompact type) we introduced an analogue of hyperbolic quasigeodesics, which we call *Morse quasigeodesics*. Morse quasigeodesics are defined relatively to a certain face τ_{mod} of the model spherical face σ_{mod} of X. In addition to the quasiisometry constants L, A, τ_{mod}-Morse quasigeodesics come equipped with two other parameters, a positive number D and a *Weyl-convex* subset Θ of the *open star* of τ_{mod} in the modal spherical chamber σ_{mod}. In [KLP1, KLP5, KLP2] we also defined τ_{mod}-Morse maps $Y \to X$ from Gromov-hyperbolic spaces to symmetric spaces. These maps are defined by the property that they send geodesics to *uniformly τ_{mod}-Morse quasigeodesics*, i.e. τ_{mod}-Morse quasigeodesics with a fixed set of parameters, (Θ, D, L, A).

The main result of this paper is a *local* characterization of Morse quasigeodesics in X:

Theorem 1.1 (Local-to-global principle for Morse quasigeodesics). *For L, A, Θ, Θ', D there exist S, L', A', D' such that every S-local (Θ, D, L, A)-local Morse quasigeodesic in X is a (Θ', D', L', A')–Morse quasigeodesic.*

Here S-locality of a certain property of a map means that this property is satisfied for restrictions of this map to subintervals of length S. We refer to Definition 3.34 and Theorem 3.34 for the details. Based on this principle, we prove in Section 3.7 a local-to-global principle for Morse maps from hyperbolic metric spaces to symmetric spaces.

We prove several consequences of these local-to-global principles:

1. The *structural stability* of Morse subgroups of G, generalizing Sullivan's Structural Stability Theorem in rank one [Su] (see also [KKL] for a detailed proof); see Theorems 4.4 and 4.6.

While structural stability for Anosov subgroups was known earlier (Labourie, Guichard-Wienhard), our method is more general and applies to a wider class of discrete subgroups, see [KL4].

Theorem 1.2 (Openness of the space of Morse actions). *For a word hyperbolic group Γ, the subset of τ_{mod}-Morse actions is open in* $\mathrm{Hom}(\Gamma, G)$.

Theorem 1.3 (Structural stability). *Let Γ be word hyperbolic. Then for τ_{mod}-Morse actions $\rho : \Gamma \curvearrowright X$, the boundary embedding $\alpha_\rho : \partial_\infty \Gamma \to \mathrm{Flag}(\tau_{mod})$ depends continuously on the action ρ.*

In particular, actions sufficiently close to a faithful Morse action are again discrete and faithful. We supplement this structural stability theorem with a stability theorem on *domains of proper discontinuity*, Theorem 4.8.

2. The locality of the Morse property implies that Morse subgroups are algorithmically recognizable; Section 4.3:

Theorem 1.4 (Semidecidability of Morse property of group actions). *Let Γ be word hyperbolic. Then there exists an algorithm whose inputs are homomorphisms $\rho : \Gamma \to G$ (defined on generators of Γ) and which terminates if and only if ρ defines a τ_{mod}-Morse action $\Gamma \curvearrowright X$.*

If the action is not Morse, the algorithm runs forever. Note that in view of [K2], there are no algorithms (in the sense of BSS computability) which would recognize if a representation $\Gamma \to \mathrm{Isom}(\mathbb{H}^3)$ is *not geometrically finite*.

3. We illustrate our techniques by constructing Morse-Schottky actions of free groups on higher rank symmetric spaces; Section 4.2. Unlike all previously known constructions, our proof does not rely on ping-pong arguments, but is purely geometric and proceeds by constructing equivariant quasi-isometric embeddings of trees. The key step is the observation that a certain local *straightness* property for sufficiently spaced sequences of points in the symmetric space implies the global Morse property. This observation is also at the heart of the proof of the local-to-global principle for Morse actions.

Since [KLP1] was originally posted in 2014, several improvements on the material of section 7 of [KLP1] and, hence, of the present paper were made:

(a) Different forms of Combination Theorems for Anosov subgroups were proven in [DKL, DK1, DK2] in the papers by the 1st and the 2nd author and, subsequently, by the 1st author and Subhadip Dey. The first one was a generalization of the technique in section 4.2 the present paper, but the other two generalizations are based on a form of the ping-pong argument.

(b) Explicit estimates in the local-to-global principle for Morse quasigeodesics and, hence, Morse embeddings, were obtained by Max Riestenberg in [1]. Riestenberg's estimates are based on replacing certain limiting arguments used in the present paper with differential-geometric and Lie-theoretic arguments.

Organization of the paper.

The notions of Morse quasigeodesics and actions are discussed in detail in section 3. In that section, among other things, we establish local-to-global principles for Morse quasigeodesics.

In section 4 we apply local-to-global principles to discrete subgroups of Lie groups: We show that Morse actions are structurally stable and algorithmically recognizable. We also construct Morse-Schottky actions of free groups on symmetric spaces. In section 5 (the appendix to the paper) we prove further properties of Morse quasigeodesics that we found to be useful in our work.

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2 Preliminaries

2.1 Basic notions of geometry of symmetric spaces

Throughout the paper we will be using definitions, notations and results of our earlier work.

We refer the reader to our earlier papers, e.g. [KLP4, KLP5, KLP2, KL1, KL2] for the various notions related to symmetric spaces, such as *polyhedral Finsler metrics* on symmetric spaces ([KL1]), the *opposition involution* ι of σ_{mod}, *model faces* τ_{mod} of σ_{mod} and the associated τ_{mod}-flag manifolds $\mathrm{Flag}(\tau_{mod})$ (sections 2.2.2 and 2.2.3 of [KLP5]), *type map* $\theta : \partial_\infty X \to \sigma_{mod}$, open Schubert cells $C(\tau) \subset \mathrm{Flag}(\tau_{mod})$ (section 2.4 of [KLP5]), Δ-valued distances d_Δ on X (section 2.6 of [KLP5]), Θ-regular geodesic segments (see §2.5.3 of [KLP5]), *parallel sets*, *stars*, *open stars* and Θ-stars, $\mathrm{st}(\tau)$, $\mathrm{ost}(\tau)$, and $\mathrm{st}_\Theta(\tau)$, *Weyl sectors* $V(x,\tau)$ (section 2.4 of [KLP5]), *Weyl cones* $V(x, \mathrm{st}(\tau))$ and Θ-cones $V(x, \mathrm{st}_\Theta(\tau))$, *diamonds* $\diamondsuit_{\tau_{mod}}(x, y)$ and Θ-diamonds $\diamondsuit_\Theta(x, y)$ (section 2.5 of [KLP5]), *τ_{mod}-regular sequences and groups* (section 4.2 of [KLP5]), τ_{mod}-convergence subgroups, flag-convergence, the Finsler interpretation of flag-convergence (see [KL1, §4.5 and 5.2] and [KLP5]), τ_{mod}-limit sets $\Lambda_{\tau_{mod}}(\Gamma) \subset \mathrm{Flag}(\tau_{mod})$ (section 4.5 of [KLP5]), visual limit set (page 4 of [KLP5]), uniformly τ_{mod}-regular sequences and subgroups (section 4.6 of [KLP5]), Morse subgroups (section 5.4 of [KLP5]) and, more generally, Morse quasigeodesics and Morse maps (Definitions 5.31, 5.33 of [KLP2]), antipodal limit sets (Definition. 5.1 of [KLP5]) and antipodal maps to flag-manifolds (Definition 6.11 of [KLP2]).

In the paper we will be frequently using convexity of Θ-cones in X:

Proposition 2.1 (Proposition 2.10 in [KLP5]). *For every Weyl-convex subset* $\Theta \subset \mathrm{st}(\tau_{mod})$, *for every $x \in X$ and $\tau \in \mathrm{Flag}(\tau_{mod})$, the cone $V(x, \mathrm{st}_\Theta(\tau)) \subset X$ is convex.*

2.2 Standing notation and conventions

- We will use the notation X for a symmetric space of *noncompact type*, G for a semisimple Lie group acting isometrically and transitively on X, and K is a maximal compact subgroup of G, so that X is diffeomorphic to G/K. We will assume that G is commensurable with the isometry group $\mathrm{Isom}(X)$ in the sense that we allow finite kernel and cokernel for the natural map $G \to \mathrm{Isom}(X)$. In particular, the image of G in $\mathrm{Isom}(X)$ contains the identity component $\mathrm{Isom}(X)_o$.
- We let $\tau_{mod} \subseteq \sigma_{mod}$ be a fixed ι-invariant face type.
- We will use the notation $x_n \xrightarrow{f} \tau \in \mathrm{Flag}(\tau_{mod})$ for the *flag-convergence* of a τ_{mod}-regular sequence $x_n \in X$ to a simplex $\tau \in \mathrm{Flag}(\tau_{mod})$.
- We will be using the notation Θ, Θ' for an ι-invariant, compact, Weyl-convex (see Definition 2.7 in [KLP5]) subset of the open star $\mathrm{ost}(\tau_{mod}) \subset \sigma_{mod}$.
- We will always assume that $\Theta < \Theta'$, meaning that $\Theta \subset \mathrm{int}(\Theta')$.
- Constants $L, A, D, \epsilon, \delta, l, a, s, S$ are meant to be always strictly positive and $L \geqslant 1$.

2.3 ζ-angles

We fix as auxiliary datum a ι-invariant type $\zeta = \zeta_{mod} \in \mathrm{int}(\tau_{mod})$. (We will omit the subscript in ζ_{mod} in order to avoid cumbersome notation for ζ-angles.) For a simplex $\tau \subset \partial_\infty X$ of type τ_{mod}, i.e. $\tau \in \mathrm{Flag}(\tau_{mod})$, we define $\zeta(\tau) \in \tau$ as the ideal point of type ζ_{mod}. Given two such simplices $\tau_\pm \in \mathrm{Flag}(\tau_{mod})$ and a point $x \in X$, define the ζ-*angles*

$$\angle_x^\zeta(\tau_-, \tau_+) = \angle_x^\zeta(\tau_-, \xi_+) := \angle_x(\xi_-, \xi_+), \tag{2.2}$$

where $\xi_\pm = \zeta(\tau_\pm)$.

Similarly, define the ζ-*Tits angle*

$$\angle_{Tits}^\zeta(\tau_-, \tau_+) = \angle_{Tits}^\zeta(\tau_-, \xi_+) := \angle_x(\xi_-, \xi_+), \tag{2.3}$$

where x belongs to a flat $F \subset X$ such that $\tau_-, \tau_+ \subset \partial_{Tits} F$. Then simplices $\tau_\pm$ (of the same type) are antipodal iff

$$\angle_{Tits}^\zeta(\tau_-, \tau_+) = \pi$$

for some, equivalently, every, choice of ζ as above.

Remark 2.4. We observe that the ideal points $\zeta_\pm$ are opposite, $\angle_{Tits}(\zeta_-, \zeta_+) = \pi$, if and only if they can be seen under angle $\simeq \pi$ (i.e., close to π) from some point in X. More precisely, there exists $\epsilon(\zeta_{mod})$ such that:

If $\angle_x(\zeta_-, \zeta_+) > \pi - \epsilon(\zeta_{mod})$ for some point x then $\zeta_\pm$ are opposite.

This follows from the angle comparison $\angle_x(\zeta_-, \zeta_+) \leqslant \angle_{Tits}(\zeta_-, \zeta_+)$ and the fact that the Tits distance between ideal points of the fixed type ζ_{mod} takes only finitely many values.

For a τ_{mod}-regular unit tangent vector $v \in TX$ we denote by $\tau(v) \subset \partial_\infty X$ the unique simplex of type τ_{mod} such that ray ρ_v with the initial direction v represents an ideal point in $\mathrm{ost}(\tau(v))$. We put $\zeta(v) = \zeta(\tau(v))$. Note that $\zeta(v)$ depends continuously on v.

For a τ_{mod}-regular segment xy in X we let $\tau(xy) = \tau(v)$, where v is the unit vector tangent to xy.

Then, for a τ_{mod}-regular segments xy, xz and $\tau \in \mathrm{Flag}(\tau_{mod})$, we define the ζ-angles

$$\angle_x^\zeta(y, \tau) = \angle_x^\zeta(\tau(xy), \tau), \quad \angle_x^\zeta(y, z) = \angle_x^\zeta(\tau(xy), \tau(xz))$$

Thus, the ζ-angle depends not on y, z but rather on the simplices $\tau(xy), \tau(xz)$. These ζ-angles will play the role of angles the between diamonds $\diamondsuit_{\tau_{mod}}(x, y)$ and $\diamondsuit_{\tau_{mod}}(x, z)$, meeting at x. Note that if X has rank 1, then the ζ-angles are just the ordinary Riemannian angles.

2.4 Distances to parallel sets versus angles

In this section we collect some geometric facts regarding parallel sets in symmetric spaces, primarily dealing with estimation of distances from points in X to parallel sets.

Remark 2.5. The constants and functions in this section are not explicit and their existence is proven by compactness arguments. For explicit computations here and in Theorem 3.18, we refer the reader to the PhD thesis of ...

We first prove a lemma (Lemma 2.6) which strengthens Corollary 2.46 of [KLP5].

Lemma 2.6. *Suppose that $\tau_\pm$ are antipodal simplices in $\partial_{Tits} X$. Then every geodesic ray γ asymptotic to a point $\xi \in ost(\tau_+)$, is strongly asymptotic to a geodesic ray in $P(\tau_-, \tau_+)$.*

Proof. If ξ belongs to the interior of the simplex τ_+, then the assertion follows from Corollary 2.46 of [KLP5]:

Weyl sectors $V(x_1, \tau)$ and $V(x_2, \tau)$ are strongly asymptotic if and only if x_1 and x_2 lie in the same horocycle at τ.

We now consider the general case. Suppose, that ξ belongs to an open simplex $\mathrm{int}(\tau')$, such that τ is a face of τ'. Then there exists an apartment $a \subset \partial_{Tits} X$ containing both ξ (and, hence, τ' as well as τ) and the simplex τ_-. Let $F \subset X$ be the maximal flat with $\partial_\infty F = a$. Then F contains a geodesic asymptotic to points in τ_- and τ_+. Therefore, F is contained in $P(\tau_-, \tau_+)$. On the other hand, by the same Corollary 2.46 of [KLP5], applied to the simplex τ', we conclude that γ is strongly asymptotic to a geodesic ray in F. $\square$

The following lemma provides a quantitative strengthening of the conclusion of Lemma 2.6:

Lemma 2.7. *Let Θ be a compact subset of $ost(\tau_+)$. Then those rays $x\xi$ with $\theta(\xi) \in \Theta$ are uniformly strongly asymptotic to $P(\tau_-, \tau_+)$, i.e. $d(\cdot, P(\tau_-, \tau_+))$ decays to zero along them uniformly in terms of $d(x, P(\tau_-, \tau_+))$ and Θ.*

Proof. Suppose that the assertion of lemma is false, i.e., there exists $\epsilon > 0$, a sequence $T_i \in \mathbb{R}_+$ diverging to infinity, and a sequence of rays $\rho_i = x_i\xi_i$ with $\xi_i \in \Theta$ and $d(x_i, P(\tau_-, \tau_+)) \leqslant d$, so that

$$d(y, P(\tau_-, \tau_+)) \geqslant \epsilon, \forall y \in \rho([0, T_i]). \tag{2.8}$$

Using the action of the stabilizer of $P(\tau_-, \tau_+)$, we can assume that the points x_i belong to a certain compact subset of X. Therefore, the sequence of rays $x_i\xi_i$ subconverges to a ray $x\xi$ with $d(x, P(\tau_-, \tau_+)) \leqslant d$ and $\xi \in \Theta$. The inequality (2.8) then implies that the entire limit ray $x\xi$ is contained outside of the open ϵ-neighborhood of the parallel set $P(\tau_-, \tau_+)$. However, in view of Lemma 2.6, the ray $x\xi$ is strongly asymptotic to a geodesic in $P(\tau_-, \tau_+)$. Contradiction. □

We next relate distances from points $x \in X$ to parallel sets and the ζ-angles at x. Suppose that the simplices $\tau_\pm$, equivalently, the ideal points $\zeta_\pm = \zeta(\tau_\pm)$ (see section 2.3), are opposite. Then

$$\angle^{\zeta}_{x}(\tau_-, \tau_+) = \angle_x(\zeta_-, \zeta_+) = \pi$$

if and only if x lies in the parallel set $P(\tau_-, \tau_+)$. Furthermore, $\angle^{\zeta}_{x}(\tau_-, \tau_+) \simeq \pi$ if and only if x is close to $P(\tau_-, \tau_+)$, and both quantities control each other near the parallel set. More precisely:

Lemma 2.9. *(i) If $d(x, P(\tau_-, \tau_+)) \leqslant d$, then $\angle^{\zeta}_{x}(\tau_-, \tau_+) \geqslant \pi - \epsilon(d)$ with $\epsilon(d) \to 0$ as $d \to 0$.*

(ii) For sufficiently small ϵ, $\epsilon \leqslant \epsilon'(\zeta_{mod})$, we have: The inequality $\angle^{\zeta}_{x}(\tau_-, \tau_+) \geqslant \pi - \epsilon$ implies that $d(x, P(\tau_-, \tau_+)) \leqslant d(\epsilon)$ for some function $d(\epsilon)$ which converges to 0 as $\epsilon \to 0$.

Proof. The intersection of parabolic subgroups $P_{\tau_-} \cap P_{\tau_+}$ preserves the parallel set $P(\tau_-, \tau_+)$ and acts transitively on it. Compactness and the continuity of $\angle_{\cdot}(\zeta_-, \zeta_+)$ therefore imply that $\pi - \angle_{\cdot}(\zeta_-, \zeta_+)$ attains on the boundary of the tubular r-neighborhood of $P(\tau_-, \tau_+)$ a strictly positive maximum and minimum, which we denote by $\phi_1(r)$ and $\phi_2(r)$. Furthermore, $\phi_i(r) \to 0$ as $r \to 0$. We have the estimate:

$$\pi - \phi_1(d(x, P(\tau_-, \tau_+))) \leqslant \angle_x(\zeta_-, \zeta_+) \leqslant \pi - \phi_2(d(x, P(\tau_-, \tau_+)))$$

The functions $\phi_i(r)$ are (weakly) monotonically increasing. This follows from the fact that, along rays asymptotic to ζ_- or ζ_+, the angle $\angle_{\cdot}(\zeta_-, \zeta_+)$ is monotonically increasing and the distance $d(\cdot, P(\tau_-, \tau_+))$ is monotonically decreasing. The estimate implies the assertions. □

The control of $d(\cdot, P(\tau_-, \tau_+))$ and $\angle_{\cdot}(\zeta_-, \zeta_+)$ "spreads" along the Weyl cone $V(x, \mathrm{st}(\tau_+))$, since the latter is asymptotic to the parallel set $P(\tau_-, \tau_+)$. Moreover, the control improves, if one enters the cone far into a τ_{mod}-regular direction. More precisely:

Lemma 2.10. *Let $y \in V(x, \mathrm{st}_\Theta(\tau_+))$ be a point with $d(x, y) \geqslant l$.*

(i) If $d(x, P(\tau_-, \tau_+)) \leqslant d$, then

$$d(y, P(\tau_-, \tau_+)) \leqslant D'(d, \Theta, l) \leqslant d$$

with $D'(d, \Theta, l) \to 0$ as $l \to +\infty$.

(ii) For sufficiently small ϵ, $\epsilon \leqslant \epsilon'(\zeta_{mod})$, we have: If $\angle_x(\zeta_-, \zeta_+) \geqslant \pi - \epsilon$, then

$$\angle_y(\zeta_-, \zeta_+) \geqslant \pi - \epsilon'(\epsilon, \Theta, l) \geqslant \pi - \epsilon(d(\epsilon))$$

with $\epsilon'(\epsilon, \Theta, l) \to 0$ as $l \to +\infty$.

Proof. The distance from $P(\tau_-, \tau_+)$ takes its maximum at the tip x of the cone $V(x, \mathrm{st}(\tau_+))$, because it is monotonically decreasing along the rays $x\xi$ for $\xi \in \mathrm{st}(\tau_+)$. This yields the right-hand bounds d and, applying Lemma 2.9 twice, $\epsilon(d(\epsilon))$.

Those rays $x\xi$ with uniformly τ_{mod}-regular type $\theta(\xi) \in \Theta$ are uniformly strongly asymptotic to $P(\tau_-, \tau_+)$, i.e. $d(\cdot, P(\tau_-, \tau_+))$ decays to zero along them uniformly in terms of d and Θ, see Lemma 2.7. This yields the decay $D'(d, \Theta, l) \to 0$ as $l \to +\infty$. The decay of ϵ' follows by applying Lemma 2.9 again. □

3 Morse maps

In this section we investigate the Morse property of sequences and maps. The main aim of this section is to establish a local criterion for being Morse. To do so we introduce a local notion of *straightness* for sequences of points in X. Morse sequences are in general not straight, but they become straight after suitable modification, namely by sufficiently coarsifying them and then passing to the sequence of successive midpoints. Conversely, the key result is that sufficiently spaced straight sequences are Morse. We conclude that there is a local-to-global characterization of the Morse property.

3.1 Morse quasigeodesics

Definition 3.1 (Morse quasigeodesic). A (Θ, D, L, A)-*Morse quasigeodesic* in X is an (L, A)-quasigeodesic $p : I \to X$ (defined on an interval $I \subset \mathbb{R}$) such that for all $t_1, t_2 \in I$ the subpath $p|_{[t_1,t_2]}$ is D-close to a Θ-diamond $\diamondsuit_\Theta(x_1, x_2)$ with $d(x_i, p(t_i)) \leqslant D$.

We will refer to a quadruple (Θ, D, L, A) as a *Morse datum* and abbreviate $M = (\Theta, D, L, A)$. Set $M + D' = (\Theta, D + D', L, A + 2D')$. We say that M contains Θ if M has the form (Θ, D, L, A) for some $D \geqslant 0, L \geqslant 1, A \geqslant 0$.

The following lemma is immediate from the definiton of a M-Morse quasigeodesic.

Lemma 3.2 (Perturbation lemma). *If p, p' are paths in X such that p is M-Morse and $d(p, p') \leqslant D'$ then p' is $M + D'$-Morse.*

A Morse quasigeodesic p is called a *Morse ray* if its domain is a half-line. If $I = \mathbb{R}$ then a Morse quasigeodesic is called a *Morse quasiline*.

Morse quasirays do in general not converge at infinity (in the visual compactification of X), but they τ_{mod}-converge at infinity. This is a consequence of:

Lemma 3.3 (Conicality). *Every Morse quasiray $p : [0, \infty) \to X$ is uniformly Hausdorff close to a subset of a cone $V(p(0), \mathrm{st}_\Theta(\tau))$ for a unique simplex τ of type τ_{mod}.*

Proof. The subpaths $p|_{[0,t_0]}$ are uniformly Hausdorff close to Θ-diamonds. These subconverge to a cone $V(x, \mathrm{st}_\Theta(\tau))$ x uniformly close to $p(0)$ and τ a simplex of type τ_{mod}. This establishes the existence. Since $p(n) \xrightarrow{f} \tau$, the uniqueness of τ follows from the uniqueness of τ_{mod}-limits, see [KLP5, Lemma 4.23]. $\square$

Definition 3.4 (End of Morse quasiray). We call the unique simplex given by the previous lemma the *end* of the Morse quasiray $p : [0, \infty) \to X$ and denote it by

$$p(+\infty) \in \mathrm{Flag}(\tau_{mod}).$$

Hausdorff close Morse quasirays have the same end by Lemma 3.3. In section 3.3 we will prove uniform continuity of ends of Morse quasirays with respect to the topology of *coarse convergence* of quasirays.

3.2 Morse maps

We now turn to *Morse maps* with more general domains (than just intervals).

Definition 3.5. Let Y be a Gromov-hyperbolic geodesic metric space. A map $f : Y \to X$ is called M-Morse if it sends geodesics in Y to M-Morse quasigeodesics.

Thus, every Morse map is a quasiisometric embedding. While this definition makes sense for general metric spaces, in [KLP2] we proved that the domain of a Morse map is necessarily hyperbolic.

More generally, one can define Morse maps on *quasigeodesic metric spaces*:

Definition 3.6 (Quasigeodesic metric space). A metric space Z is called (l, a)-*quasigeodesic* if all pairs of points in Y can be connected by (l, a)-quasigeodesics. A space is called *quasigeodesic* if it is (l, a)-quasigeodesic for some pair of parameters l, a.

Every quasigeodesic space is quasiisometric to a geodesic metric space. Namely, if Z is (λ, α)-quasigeodesic space then it is quasiisometric to its $(\lambda + \alpha)$-Rips complex. The quasigeodesic spaces considered in this paper are discrete groups equipped with word metrics.

Definition 3.7 (Morse embedding). Let (Θ, D, L, A) be a Morse datum. An (Θ, D, L, A, l, a)-*Morse embedding* (or a map) from an (l, a)-quasigeodesic space Z into X is a map $f : Z \to X$ which sends (l, a)-quasigeodesics in Z to (Θ, D, L, A)-Morse quasigeodesics in X.

Of course, every (l, a)-quasigeodesic metric space is also (l', a')-quasigeodesic space for any

$l' \geqslant l, a' \geqslant a$. The next lemma shows that this choice of quasigeodesic constants is essentially irrelevant.

Lemma 3.8. *Let $f : Z \to X$ be a map from a Gromov-hyperbolic (l, a)-quasigeodesic space Z. If f is $M = (\Theta, D, L, A, l, a)$-Morse then for any (l', a'), it sends (l', a')-quasigeodesics in Z to $M' = (\Theta, D', L', A')$-Morse quasigeodesics in X. Here the datum M' depends only on M, l', a' and the hyperbolicity constant δ of Z.*

Proof. This is a consequence of the definition of Morse quasigeodesics, and the Morse Lemma applied to Z. ☐

Notice that the parameter Θ in the Morse datum M' is the same as in M. Hence, we arrive to

Definition 3.9. A map $f : Z \to X$ of a quasigeodesic hyperbolic space Z is called Θ-*Morse* if it sends uniform quasigeodesics in Z to Θ-Morse uniform quasigeodesics in X.

This notion depends only on the quasi-isometry class of Z, i.e. the precomposition of a Θ-Morse embedding with a quasi-isometry is again Θ-Morse. For this to be true we have to require control on the images of quasigeodesics of arbitrarily bad (but uniform) quality.

Let Γ be a hyperbolic group with fixed a finite generating set S, and let Y be the Cayley graph of Γ with respect to S. For $x \in X$, an isometric action $\Gamma \curvearrowright X$ determines the *orbit map* $o_x : \Gamma \to \Gamma x \subset X$. Every such map extends to the Cayley graph Y of Γ, sending edges to geodesics in X.

Definition 3.10. An isometric action $\Gamma \curvearrowright X$ or a representation $\rho : \Gamma \to G$, is called M-Morse (with respect to a base-point $x \in X$) if the (extended) orbit map $o_x : Y \to X$ is M-Morse. Similarly, a subgroup $\Gamma < G$ is Morse if the inclusion homomorphism $\Gamma \hookrightarrow G$ is Morse.

The Morse property of an action and the parameter Θ, of course, does not depend on the choice of a generating set of Γ and a base-point x, but the triple (D, L, A) does. Thus, it makes sense to talk about a Θ-Morse and τ_{mod}-Morse actions of hyperbolic groups, where $\Theta \subset \mathrm{ost}(\tau_{mod})$. In [KLP5, KLP2, KL1] we gave many alternative definitions of Morse actions, including the equivalence of this definition to the notion of Anosov subgroups.

3.3 Continuity at infinity

Let X, Y be proper metric spaces. We fix a base point $y \in Y$.

Definition 3.11. A sequence of maps $f_n : Y \to X$ is said to *coarsely converge* to a map $f : Y \to X$ if there exists $C < \infty$ such that for every R there exists $N = N(C, R)$ for which

$$d(f_n|_B, f|_B) \leqslant C,$$

where $B = B(y, R)$.

Note the difference of this definition with the notion of uniform convergence on compacts: Since we are working in the coarse setting, requiring the distance between maps to be less than ϵ close to zero is pointless.

In view of the Arzela–Ascoli theorem, the space of (L, A)-coarse Lipschitz maps $Y \to X$ sending y to a fixed bounded subset of X, is coarsely sequentially compact: Every sequence contains a coarsely converging subsequence.

In the next lemma we assume that Y is a geodesic δ-hyperbolic space and X is a symmetric space of noncompact type. The lemma itself is an immediate consequence of the perturbation lemma, Lemma 3.2.

Lemma 3.12. *Suppose that $p_n : \mathbb{R}_+ \to X$ is a sequence of M-Morse rays which coarsely converges to a map $p : \mathbb{R}_+ \to X$. Then p is M'-Morse, where $M' = M + C$ and the constant C is the one appearing in the definition of coarse convergence.*

In particular, a coarse limit of a sequence of (uniformly) Morse quasigeodesics is again Morse.

For the next lemma, we equip the flag manifold $\mathrm{F} = \mathrm{Flag}(\tau_{mod})$ with some background metric d_{F}.

Lemma 3.13. *Suppose that $p_n : \mathbb{R}_+ \to X$ is a sequence of M-Morse rays coarsely converging to a M-Morse ray $p : \mathbb{R}_+ \to X$. Then the sequence $\tau_n := p_n(\infty)$ of ends of the quasirays p_n converges to $\tau = p(\infty)$. Moreover, the latter convergence is uniform in the following sense. For every $\epsilon > 0$ there exists n_0 depending only on M and C and $N(R, C)$ (appearing in Definition 3.11) such that for all $n \geqslant n_0$, $d_{\mathrm{F}}(\tau_n, \tau) \leqslant \epsilon$.*

Proof. Suppose that the claim is false. Then in view of coarse compactness of the space of M-Morse maps sending y to a fixed compact subset of X, there exists a sequence (p_n) as in the lemma, coarsely converging to p, such that the sequence $p_n(\infty) = \tau_n$ converges to $\tau' \neq p(\infty) = \tau$. By the coarse convergence $p_n \to p$, there exists $C < \infty$ and a sequence $t_n \to \infty$ such that $d(p_n(t_n), p(t_n)) \leqslant C$. By the definition of Morse quasigeodesics, there exists a sequence of cones $V(x_n, \mathrm{st}(\tau_n))$ (with x_n in a bounded subset $B \subset X$) such that the image of p_n is contained in the D-neighborhood of $V(x_n, \mathrm{st}(\tau_n))$. Thus, the sequence $(p_n(t_n))$ flag-converges to τ', while $(p(t_n))$ flag-converges to τ. According to [KLP5, Lemma 4.23], altering a sequence by a uniformly bounded amount, does not change the flag-limit. Therefore, the sequence $(p(t_n))$ also flag-converges to τ'. Hence, $\tau = \tau'$. A contradiction. $\square$

3.4 A Morse Lemma for straight sequences

In order to motivate the results of this section we recall the following *sufficient condition* for a piecewise-geodesic path in a Hadamard manifold Y of curvature $\leqslant -1$ to be quasigeodesic (see e.g. [KaLi]):

Proposition 3.14. *Suppose that c is a piecewise-geodesic path in Y whose angles at the vertices are $\geqslant \alpha > 0$ and whose edges are longer than L, where α and L satisfy*

$$\cosh(L/2)\sin(\alpha/2) \geqslant \nu > 1. \tag{3.15}$$

Then c is an $(L(\nu), A(\nu))$-quasigeodesic.

By considering c with vertices on a horocycle in the hyperbolic plane, one see that the inequality in this proposition is sharp.

Corollary 3.16. *If L is sufficiently large and α is sufficiently close to π then c is (uniformly) quasigeodesic.*

In higher rank, we do not have an analogue of the inequality (3.15), instead, we will be generalizing the corollary. However, *angles* in the corollary will be replaced with *ζ-angles*. We will show (in a String of Diamonds Theorem, theorem 3.30) that if a piecewise-geodesic path c in X has sufficiently long edges and ζ-angles between consecutive segments sufficiently close to π, then c is M-Morse for a suitable Morse datum.

In the following, we consider finite or infinite sequences (x_n) of points in X.

Definition 3.17 (Straight and spaced sequence). We call a sequence (x_n) (Θ, ϵ)-*straight* if the segments $x_n x_{n+1}$ are Θ-regular and

$$\angle^{\zeta}_{x_n}(x_{n-1}, x_{n+1}) \geqslant \pi - \epsilon$$

for all n. We call it l-*spaced* if the segments $x_n x_{n+1}$ have length $\geqslant l$.

Note that every straight sequence can be extended to a biinfinite straight sequence.

Straightness is a local condition. The goal of this section is to prove the following local-to-global result asserting that sufficiently straight and spaced sequences satisfy a higher rank version of the Morse Lemma (for quasigeodesics in hyperbolic space).

Theorem 3.18 (Morse Lemma for straight spaced sequences). *For Θ, Θ', δ there exist l, ϵ such that:*

Every (Θ, ϵ)-straight l-spaced sequence (x_n) is δ-close to a parallel set $P(\tau_-, \tau_+)$ with simplices $\tau_\pm$ of type τ_{mod}, and it moves from τ_- to τ_+ in the sense that its nearest point projection $\bar{x}_n$ to $P(\tau_-, \tau_+)$ satisfies

$$\bar{x}_{n\pm m} \in V(\bar{x}_n, \mathrm{st}_{\Theta'}(\tau_\pm)) \tag{3.19}$$

for all n and $m \geqslant 1$.

Remark 3.20 (Global spacing). 1. As a corollary of this theorem, we will show that straight spaced sequences are quasigeodesic:

$$d(x_n, x_{n+m}) \geqslant clm - 2\delta$$

with a constant $c = c(\Theta') > 0$. See Corollary 3.29. In particular, by interpolating the sequence (x_n) via geodesic segments we obtain a Morse quasigeodesic in X.

2. Theorem 3.18 is a higher-rank generalization of two familiar facts from geometry of Gromov-hyperbolic geodesic metric spaces: The fact that local quasigeodesics (with suitable parameters) are global quasigeodesics and the Morse lemma stating that quasigeodesics stay uniformly close to geodesics. In the higher rank, quasigeodesics, of course, need not be close to geodesics, but, instead (under the straightness assumption), are close to diamonds/Weyl cones/parallel sets.

3. One can obviously strengthen the Corollary 3.16 by stating that for each $\epsilon < \pi$ there exists $L_0(\epsilon)$ such that if $\alpha \geqslant \pi - \epsilon$ and $L \geqslant L_0(\epsilon)$ then c is a uniform quasigeodesic in X. A similar strengthening is false for symmetric spaces of rank $\geqslant 2$. For instance, when $W \cong S_3$ and $\epsilon = 2\pi/3$, then no matter what Θ, Θ' and l are, the conclusion of Theorem 3.18 fails already for sequences contained in a single flat.

In order to prove the theorem, we start by considering half-infinite sequences and prove that they keep moving away from an ideal simplex of type τ_{mod} if they do so initially.

Definition 3.21 (Moving away from an ideal simplex). Given a face $\tau \subset \partial_{Tits} X$ of type τ_{mod} and distinct points $x, y \in X$, define the angle

$$\angle_x^\zeta(\tau, y) := \angle_x(z, y)$$

where z is a point (distinct from x) on the geodesic ray $x\xi$, where $\xi \in \tau$ is the point of type ζ.

We say that a sequence (x_n) *moves ϵ-away* from a simplex τ of type τ_{mod} if

$$\angle_{x_n}^\zeta(\tau, x_{n+1}) \geqslant \pi - \epsilon$$

for all n.

Lemma 3.22 (Moving away from ideal simplices). *For small ϵ and large l, $\epsilon \leqslant \epsilon_0$ and $l \geqslant l(\epsilon, \Theta)$, the following holds:*

If the sequence $(x_n)_{n \geqslant 0}$ is (Θ, ϵ)-straight l-spaced and if

$$\angle_{x_0}^\zeta(\tau, x_1) \geqslant \pi - 2\epsilon,$$

then (x_n) moves ϵ-away from τ.

Proof. By Lemma 2.10(ii), the unit speed geodesic segment $c : [0, t_1] \to X$ from $p(0)$ to $p(1)$ moves $\epsilon(d(2\epsilon))$-away from τ at all times, and $\epsilon'(2\epsilon, \Theta, l)$-away at times $\geqslant l$, which includes the final time t_1. For $l(\epsilon, \Theta)$ sufficiently large, we have $\epsilon'(2\epsilon, \Theta, l) \leqslant \epsilon$. Then c moves ϵ-away from τ at time t_1, which means that $\angle_{x_1}^\zeta(\tau, x_0) \leqslant \epsilon$. Straightness at x_1 and the triangle inequality yield that again $\angle_{x_1}^\zeta(\tau, x_2) \geqslant \pi - 2\epsilon$. One proceeds by induction. ☐

Note that there do exist simplices τ satisfying the hypothesis of the previous lemma. For instance, one can extend the initial segment $x_0 x_1$ backwards to infinity and choose $\tau = \tau(x_1 x_0)$.

Now we look at *biinfinite* sequences.

We assume in the following that $(x_n)_{n\in\mathbb{Z}}$ is (Θ,ϵ)-straight l-spaced for small ϵ and large l. As a first step, we study the asymptotics of such sequences and use the argument for Lemma 3.22 to find a pair of opposite ideal simplices $\tau_\pm$ such that (x_n) moves from τ_- towards τ_+.

Lemma 3.23 (Moving towards ideal simplices). *For small ϵ and large l, $\epsilon \leqslant \epsilon_0$ and $l \geqslant l(\epsilon,\Theta)$, the following holds:*

There exists a pair of opposite simplices $\tau_\pm$ of type τ_{mod} such that the inequality

$$\angle^\zeta_{x_n}(\tau_\mp, x_{n\pm 1}) \geqslant \pi - 2\epsilon \tag{3.24}$$

holds for all n.

Proof. 1. For every n define a compact set $C_n^\mp \subset \mathrm{Flag}(\tau_{mod})$

$$C_n^\pm = \{\tau_\pm : \angle^\zeta_{x_n}(\tau_\pm, x_{n\mp 1}) \geqslant \pi - 2\epsilon\}.$$

As in the proof of Lemma 3.22, straightness at x_{n+1} implies that $C_n^- \subset C_{n+1}^-$. Hence the family $\{C_n^-\}_{n\in\mathbb{Z}}$ form a nested sequence of nonempty compact subsets and therefore have nonempty intersection containing a simplex τ_-. Analogously, there exists a simplex τ_+ which belongs to C_n^+ for all n.

2. It remains to show that the simplices τ_-, τ_+ are antipodal. Using straightness and the triangle inequality, we see that

$$\angle^\zeta_{x_n}(\tau_-, \tau_+) \geqslant \pi - 5\epsilon$$

for all n. Hence, if $5\epsilon < \epsilon(\zeta)$, then the simplices τ_-, τ_+ are antipodal in view of Remark 2.4. ☐

The pair of opposite simplices (τ_-, τ_+) which we found determines a parallel set in X. The second step is to show that (x_n) is uniformly close to it.

Lemma 3.25 (Close to parallel set). *For small ϵ and large l, $\epsilon \leqslant \epsilon(\delta)$ and $l \geqslant l(\Theta,\delta)$, the sequence (x_n) is δ-close to $P(\tau_-, \tau_+)$.*

Proof. The statement follows from the combination of the inequality (3.4) (in the second part of the proof of Lemma 3.23) and Lemma 2.9. ☐

The third and final step of the proof is to show that the nearest point projection $(\bar{x}_n)$ of (x_n) to $P(\tau_-, \tau_+)$ moves from τ_- towards τ_+.

Lemma 3.26 (Projection moves towards ideal simplices). *For small ϵ and large l, $\epsilon \leqslant \epsilon_0$ and $l \geqslant l(\epsilon,\Theta,\Theta')$, the segments $\bar{x}_n\bar{x}_{n+1}$ are Θ'-regular and*

$$\angle^\zeta_{\bar{x}_n}(\tau_-, \bar{x}_{n+1}) = \pi$$

for all n.

Proof. By the previous lemma, (x_n) is δ_0-close to $P(\tau_-, \tau_+)$ if ϵ_0 is sufficiently small and l is sufficiently large. Since x_nx_{n+1} is Θ-regular, the triangle inequality for Δ-lengths yields that the segment $\bar{x}_n\bar{x}_{n+1}$ is Θ'-regular, again if l is sufficiently large.

Let ξ_+ denote the ideal endpoint of the ray extending this segment, i.e. $\bar{x}_{n+1} \in \bar{x}_n\xi_+$. Then x_{n+1} is $2\delta_0$-close to the ray $x_n\xi_+$. We obtain that

$$\angle^{\zeta}_{Tits}(\tau_-, \xi_+) \geqslant \angle^{\zeta}_{x_n}(\tau_-, \xi_+) \simeq \angle^{\zeta}_{x_n}(\tau_-, x_{n+1}) \simeq \pi$$

where the last step follows from inequality (3.24). The discreteness of Tits distances between ideal points of fixed type ζ implies that in fact

$$\angle^{\zeta}_{Tits}(\tau_-, \xi_+) = \pi,$$

i.e. the ideal points $\zeta(\tau_-)$ and $\zeta(\xi_+)$ are antipodal. But the only simplex opposite to τ_- in $\partial_\infty P(\tau_-, \tau_+)$ is τ_+, so $\tau(\xi_+) = \tau_+$ and

$$\angle^{\zeta}_{\bar{x}_n}(\tau_-, \bar{x}_{n+1}) = \angle^{\zeta}_{\bar{x}_n}(\tau_-, \xi_+) = \pi,$$

as claimed. □

Proof of Theorem 3.18. It suffices to consider biinfinite sequences.

The conclusion of Lemma 3.26 is equivalent to $\bar{x}_{n+1} \in V(\bar{x}_n, \mathrm{st}_{\Theta'}(\tau_+))$. Combining Lemmas 3.25 and 3.26, we thus obtain the theorem for $m = 1$.

The convexity of Θ'-cones, cf. Proposition 2.1, implies that

$$V(\bar{x}_{n+1}, \mathrm{st}_{\Theta'}(\tau_+)) \subset V(\bar{x}_n, \mathrm{st}_{\Theta'}(\tau_+)),$$

and the assertion follows for all $m \geqslant 1$ by induction. □

Remark 3.27. The conclusion of the theorem implies flag-convergence $x_{\pm n} \to \tau_\pm$ as $n \to +\infty$. However, the sequences $(x_n)_{n \in \pm\mathbb{N}}$ do in general not converge at infinity, but accumulate at compact subsets of $\mathrm{st}_{\Theta'}(\tau_\pm)$.

3.5 Lipschitz retractions to straight paths

Consider a (possibly infinite) closed interval J in $\mathbb{R}$; we will assume that J has integer or infinite bounds. Suppose that $p : J \cap \mathbb{Z} \to P = P(\tau_-, \tau_+) \subset X$ is an l-separated, λ-Lipschitz, $(\Theta, 0)$-straight coarse sequence pointing away from τ_- and towards τ_+. We extend p to a piecewise-geodesic map $p : J \to P$ by sending intervals $[n, n+1]$ to geodesic segments $p(n)p(n+1)$ via affine maps. We retain the name p for the extension.

Lemma 3.28. *There exists $L = L(l, \lambda, \Theta)$ and an L-Lipschitz retraction of X to p, i.e., an L-Lipschitz map $r : X \to J$ so that $r \circ p = Id$. In particular, $p : J \cap \mathbb{Z} \to X$ is a $(\bar{L}, \bar{A})$-quasigeodesic, where $\bar{L}, \bar{A}$ depend only on l, λ, Θ.*

Proof. It suffices to prove existence of a retraction. Since P is convex in X, it suffices to construct a map $P \to J$. Pick a generic point $\xi = \xi_+ \in \tau_+$ and let $b_\xi : P \to \mathbb{R}$ denote the Busemann function normalized so that $b_\xi(p(z)) = 0$ for some $z \in J \cap \mathbb{Z}$. Then the Θ-regularity assumption on p implies that the slope of the piecewise-linear function $b_\xi \circ p : J \to \mathbb{R}$ is strictly positive, bounded away from 0. The assumption that p is l-separated λ-Lipschitz implies that

$$l \leqslant |p'(t)| \leqslant \lambda$$

for each t (where the derivative exists). The straightness assumption on p implies that the function $h := b_\xi \circ p : J \to \mathbb{R}$ is strictly increasing. By combining these observations, we conclude that h is an L-biLipschitz homeomorphism for some $L = L(l, \lambda, \Theta)$. Lastly, we define

$$r : P \to J, \quad r = h^{-1} \circ b_\xi.$$

Since b_ξ is 1-Lipschitz, the map r is L-Lipschitz. By the construction, $r \circ p = Id$. □

Corollary 3.29. *Suppose that $p : J \cap \mathbb{Z} \to X$ is a l-spaced, λ-Lipschitz, (Θ, ϵ)-straight sequence. Pick some Θ' such that $\Theta \subset int(\Theta')$ and let $\delta = \delta(l, \Theta, \Theta', \epsilon)$ be the constant as in Theorem 3.18. Then for $L = L(l - 2\delta, \lambda + 2\delta, \Theta')$ we have:*

1. *There exists an $(L, 2\delta)$-coarse Lipschitz retraction $X \to J$.*
2. *The map p is a (Θ', D', L', A')-quasigeodesic with D', L', A' depending only on $l, \lambda, \Theta, \Theta', \epsilon$.*

Proof. The statement immediately follows the above lemma combined with Theorem 3.18. □

Reformulating in terms of piecewise-geodesic paths, we obtain

Theorem 3.30 (String of diamonds theorem). *For any pair of Weyl convex subsets $\Theta < \Theta'$ and a number $D \geqslant 0$ there exist positive numbers ϵ, S, L, A depending on the datum (Θ, Θ', D) such that the following holds.*

Suppose that c is an arc-length parameterized piecewise-geodesic path (finite or infinite) in X obtained by concatenating geodesic segments $x_i x_{i+1}$ satisfying for all i:

1. *Each segment $x_i x_{i+1}$ is Θ-regular and has length $\geqslant S$.*
2. $$\angle^{\zeta}_{x_i}(x_{i-1}, x_{i+1}) \geqslant \pi - \epsilon.$$

Then the path c is (Θ', D, L, A)-Morse.

3.6 Local Morse quasigeodesics

According to Theorem 3.30, sufficiently straight and spaced straight piecewise-geodesic paths are Morse. In this section we will now prove that, conversely, the Morse property implies straightness in a suitable sense, namely that for sufficiently spaced quadruples the associated midpoint triples are arbitrarily straight. (For the quadruples themselves this is in general not true.)

Definition 3.31 (Quadruple condition). For points $x, y \in X$ we let $\mathrm{mid}(x, y)$ denote the midpoint of the geodesic segment xy. A map $p : I \to X$ satisfies the (Θ, ϵ, l, s)-*quadruple condition* if for all $t_1, t_2, t_3, t_4 \in I$ with $t_2 - t_1, t_3 - t_2, t_4 - t_3 \geqslant s$ the triple of midpoints

$$(\mathrm{mid}(t_1, t_2), \mathrm{mid}(t_2, t_3), \mathrm{mid}(t_3, t_4))$$

is (Θ, ϵ)-straight and l-spaced.

Proposition 3.32 (Morse implies quadruple condition). *For $L, A, \Theta, \Theta', D, \epsilon, l$ exists a scale $s = s(L, A, \Theta, \Theta', D, \epsilon, l)$ such that every (Θ, D, L, A)-Morse quasigeodesic satisfies the $(\Theta', \epsilon, l, s')$-quadruple condition for every $s' \geqslant s$.*

Proof. Let $p : I \to X$ be an (L, A, Θ, D)-Morse quasigeodesic, and let $t_1, \ldots, t_4 \in I$ such that $t_2 - t_1, t_3 - t_2, t_4 - t_3 \geqslant s$. We abbreviate $p_i := p(t_i)$ and $m_i = \mathrm{mid}(p_i, p_{i+1})$.

Regarding straightness, it suffices to show that the segment $m_2 m_1$ is Θ'-regular and that $\angle^{\zeta}_{m_2}(p_2, m_1) \leqslant \frac{\epsilon}{2}$ provided that s is sufficiently large in terms of the given data.

By the Morse property, there exists a diamond $\diamondsuit_\Theta(x_1, x_3)$ such that $d(x_1, p_1), d(x_3, p_3) \leqslant D$ and $p_2 \in N_D(\diamondsuit_\Theta(x_1, x_3))$. The diamond spans a unique parallel set $P(\tau_-, \tau_+)$. (Necessarily, $x_3 \in V(x_1, \mathrm{st}_\Theta(\tau_+))$ and $x_1 \in V(x_3, \mathrm{st}_\Theta(\tau_-))$.)

We denote by $\bar{p}_i$ and $\bar{m}_i$ the projections of p_i and m_i to the parallel set.

We first observe that m_2 (and m_3) is arbitrarily close to the parallel set if s is large enough. If this were not true, a limiting argument would produce a geodesic line at strictly positive finite Hausdorff distance $\in (0, D]$ from $P(\tau_-, \tau_+)$ and asymptotic to ideal points in $\mathrm{st}_\Theta(\tau_\pm)$. However, all lines asymptotic to ideal points in $\mathrm{st}_\Theta(\tau_\pm)$ are contained in $P(\tau_-, \tau_+)$.

Next, we look at the directions of the segments $\bar{m}_2\bar{m}_1$ and $\bar{m}_2\bar{p}_2$ and show that they have the same τ-direction. Since $\bar{p}_2$ is $2D$-close to $V(\bar{p}_1, \mathrm{st}_\Theta(\tau_+))$, we have that the point $\bar{p}_1$ is $2D$-close to $V(\bar{p}_2, \mathrm{st}_\Theta(\tau_-))$, and hence also $\bar{m}_1$ is $2D$-close to $V(\bar{p}_2, \mathrm{st}_\Theta(\tau_-))$. Therefore, $\bar{p}_1, \bar{m}_1 \in V(\bar{p}_2, \mathrm{st}_{\Theta'}(\tau_-))$ if s is large enough. Similarly, $\bar{m}_2 \in V(\bar{p}_2, \mathrm{st}_{\Theta'}(\tau_+))$ and hence $\bar{p}_2 \in V(\bar{m}_2, \mathrm{st}_{\Theta'}(\tau_-))$. The convexity of Θ'-cones, see Proposition 2.1, implies that also $\bar{m}_1 \in V(\bar{m}_2, \mathrm{st}_{\Theta'}(\tau_-))$. In particular, $\angle^{\zeta}_{\bar{m}_2}(\bar{p}_2, \bar{m}_1) = 0$ if s is sufficiently large.

Since m_2 is arbitrarily close to the parallel set if s is sufficiently large, it follows by another limiting argument that $\angle^{\zeta}_{m_2}(p_2, m_1) \leqslant \frac{\epsilon}{2}$ if s is sufficiently large.

Regarding the spacing, we use that $\bar{m}_1 \in V(\bar{p}_2, \mathrm{st}_{\Theta'}(\tau_-))$ and $\bar{m}_2 \in V(\bar{p}_2, \mathrm{st}_{\Theta'}(\tau_+))$. It follows that

$$d(\bar{m}_1, \bar{m}_2) \geqslant c \cdot (d(\bar{m}_1, \bar{p}_2) + d(\bar{p}_2, \bar{m}_2))$$

with a constant $c = c(\Theta') > 0$, and hence that $d(m_1, m_2) \geqslant l$ if s is sufficiently large. $\square$

Theorem 3.18 and Proposition 3.32 tell that the Morse property for quasigeodesics is equivalent to straightness (of associated spaced sequences of points). Since straightness is a local condition, this leads to a local to global result for Morse quasigeodesics, namely that the Morse property holds globally if it holds locally up to a sufficiently large scale.

Definition 3.33 (Local Morse quasigeodesic). An S-local (Θ, D, L, A)-*Morse quasigeodesic* in X is a map $p : I \to X$ such that for all t_0 the subpath $p|_{[t_0, t_0+S]}$ is a (Θ, D, L, A)-Morse quasigeodesic.

Note that local Morse quasigeodesics are uniformly coarse Lipschitz.

Theorem 3.34 (Local-to-global principle for Morse quasigeodesics). *For L, A, Θ, Θ', D exist S, L', A', D' such that every S-local (Θ, D, L, A)-local Morse quasigeodesic in X is an (Θ', D', L', A')-Morse quasigeodesic.*

Proof. We choose an auxiliary Weyl convex subset Θ'' such that $\Theta < \Theta'' < \Theta'$.

Let $p : I \to X$ be an S-local (Θ, D, L, A)-local Morse quasigeodesic. We consider its coarsification on a (large) scale s and the associated midpoint sequence, i.e. we put $p_n^s = p(ns)$ and $m_n^s = \mathrm{mid}(p_n^s, p_{n+1}^s)$. Whereas the coarsification itself does in general not become arbitrarily straight as the scale s increases, this is true for its midpoint sequence due to Proposition 3.32. We want it to be sufficiently straight and spaced so that we can apply to it the Morse Lemma from Theorem 3.18. Therefore we first fix an auxiliary constant δ, and further auxiliary constants l, ϵ as determined by Theorem 3.18 in terms of Θ', Θ'' and δ. Then Proposition 3.32 applied to the (Θ, D, L, A)-Morse quasigeodesics $p|_{[t_0, t_0+S]}$ yields that (m_n^s) is (Θ'', ϵ)-straight and l-spaced if $S \geqslant 3s$ and the scale s is large enough depending on $L, A, \Theta, \Theta'', D, \epsilon, l$.

Now we can apply Theorem 3.18 to (m_n^s). It yields a nearby sequence $(\bar{m}_n^s)$, $d(\bar{m}_n^s, m_n^s) \leqslant \delta$, with the following property: For all $n_1 < n_2 < n_3$ the segments $\bar{m}_{n_1}^s \bar{m}_{n_3}^s$ are uniformly regular and the points $m_{n_2}^s$ are δ-close to the diamonds $\diamondsuit_{\Theta'}(\bar{m}_{n_1}^s, \bar{m}_{n_3}^s)$.

Since the subpaths $p|_{[ns,(n+1)s]}$ filling in (p_n^s) are (L, A)-quasigeodesics (because $S \geqslant s$), and it follows that for all $t_1, t_2 \in I$ the subpaths $p|_{[t_1, t_2]}$ are D'-close to Θ'-diamonds with D' depending on L, A, s.

The conclusion of Theorem 3.18 also implies a global spacing for the sequence (m_n^s), compare Remark 3.20, i.e. $d(m_n^s, m_{n'}^s) \geqslant c \cdot |n - n'|$ with a positive constant c depending on Θ', l. Hence p is a global (L', A')-quasigeodesic with L', A' depending on L, A, s, c.

Combining this information, we obtain that p is an (Θ', D', L', A')-Morse quasigeodesic for certain constants L', A' and D' depending on L, A, Θ, Θ' and D, provided that the scale S is sufficiently large in terms of the same data. $\square$

3.7 Local-to-global principle for Morse maps

We now deduce from our local-to-global result for Morse quasigeodesics, Theorem 3.34, a local-to-global result for Morse embeddings.

We restrict to the setting of maps of Gromov-hyperbolic (l, a)-quasigeodesic metric spaces Z to symmetric spaces X.

Definition 3.35 (Local Morse embedding). We call a map $f : Z \to X$ an S-local (Θ, D, L, A)-Morse map if for any (l, a)-quasigeodesic $q : I \to Z$ defined on an interval I of length $\leqslant S$ the image path $f \circ q$ is a (Θ, D, L, A)-Morse quasigeodesic in X.

Theorem 3.36 (Local-to-global principle for Morse embeddings of Gromov hyperbolic spaces). *For $l, a, L, A, \Theta, \Theta', D$ exists a scale S and a datum (D', L', A') such that every S-local (Θ, D, L, A)-Morse embedding from an (l, a)-quasigeodesic Gromov hyperbolic space into X is a (Θ', D', L', A')-Morse embedding.*

Proof. Let $f : Z \to X$ denote the local Morse embedding. It sends every (l, a)-quasigeodesic $q : I \to Z$ to an S-local (Θ, D, L, A)-Morse quasigeodesic $p = f \circ q$ in X. By Theorem 3.34, p is (L', A', Θ', D')-Morse if $S \geqslant S(l, a, L, A, \Theta, \Theta', D)$, where L', A', D' depend on the given data. $\square$

Below is a reformulation of this theorem in the case of geodesic Gromov-hyperbolic spaces.

Let Z be a δ-hyperbolic geodesic space. An R-ball $B(z, R)$ in Z need not be convex, but it is δ-quasiconvex. In particular, the restriction of the metric from Z to $B(z, R)$ results in a $(1, \delta)$-quasigeodesic metric space.

Theorem 3.37 (Local-to-global principle for Morse embeddings of geodesic spaces). *For $L, A, \Theta, \Theta', D, \delta$ exists a scale R and a datum (D', L', A') such that if Z is a δ-hyperbolic geodesic metric space and the restriction of f to any R-ball is $(\Theta, D, L, A, 1, \delta)$-Morse, then $f : Z \to X$ is (Θ', D', L', A')-Morse.*

4 Group-theoretic applications

As a consequence of the local-to-global criterion for Morse maps, in this section we establish that the Morse property for isometric group actions is an open condition. Furthermore, for two nearby Morse actions, the actions on their τ_{mod}-limit sets are also close, i.e. conjugate by an equivariant homeomorphism close to identity. In view of the equivalence of Morse property with the asymptotic properties discussed earlier, this implies structural stability for asymptotically embedded groups. Another corollary of the local-to-global result is the algorithmic recognizability of Morse actions.

We conclude the section by illustrating our technique by constructing Morse-Schottky actions of free groups on higher rank symmetric spaces.

4.1 Stability of Morse actions

We consider isometric actions $\Gamma \curvearrowright X$ of finitely generated groups.

Definition 4.1 (Morse action). We call an action $\Gamma \curvearrowright X$ Θ-*Morse* if one (any) orbit map $\Gamma \to \Gamma x \subset X$ is a Θ-Morse embedding with respect to a(ny) word metric on Γ. We call an action $\Gamma \curvearrowright X$ τ_{mod}-*Morse* if it is Θ-Morse for some τ_{mod}-Weyl convex compact subset $\Theta \subset \mathrm{ost}(\tau_{mod})$.

Remark 4.2 (Morse actions are τ_{mod}-regular and undistorted). (i) It follows immediately from the definition of Morse quasigeodesics that Θ-Morse actions are τ_{mod}-regular for the simplex type τ_{mod} determined by Θ.

(ii) Morse subgroups of G are *undistorted* in the sense that the orbit maps are quasi-isometric embeddings. In [KL1] we prove that Morse subgroups of G satisfy a stronger property: They are *coarse Lipschitz retracts* of G. This retraction property is stronger than nondistortion: Every finitely generated subgroup which is a coarse retract of G is undistorted in G, but there are examples of undistorted subgroups which are not coarse retracts. For instance, the group

$\Phi := F_2 \times F_2$ admits an undistorted embedding in the isometry group of $X = \mathbb{H}^2 \times \mathbb{H}^2$. On the other hand, pick an epimorphism $\phi : F_2 \to \mathbb{Z}$ and define the subgroup $\Gamma < \Phi$ as the kernel of the homomorphism

$$(\gamma_1, \gamma_2) \mapsto \phi(\gamma_1) - \phi(\gamma_2).$$

Then Γ is a finitely generated undistorted subgroup of Φ (see e.g. [OS, Theorem 2]), but is not finitely presented (see e.g. [BR]). Hence, $\Gamma < G = \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$ is undistorted but is not a coarse Lipschitz retract.

We denote by $\mathrm{Hom}_{\tau_{mod}}(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)$ the subset of τ_{mod}-Morse actions $\Gamma \curvearrowright X$.

By analogy with *local Morse quasigeodesics*, we define *local Morse group actions* $\rho : \Gamma \curvearrowright X$ of a hyperbolic group (with a fixed finite generating set):

Definition 4.3. An action ρ is called S-locally (Θ, D, L, A)-*locally Morse*, or (Θ, D, L, A)-*locally Morse on the scale* S, with respect to a base-point $x \in X$, if the orbit map $\Gamma \to \Gamma \cdot x \subset X$ induces an S-local (Θ, D, L, A)-local Morse embedding of the Cayley graph of Γ.

According to our local-to-global result for Morse embeddings, see Theorem 3.37, an action of a word hyperbolic group is Morse if and only if it is local Morse on a sufficiently large scale. Since this is a finite condition, it follows that the Morse property is stable under perturbation of the action:

Theorem 4.4 (Morse is open for word hyperbolic groups). *For any word hyperbolic group* Γ *the subset* $\mathrm{Hom}_{\tau_{mod}}(\Gamma, G)$ *is open in* $\mathrm{Hom}(\Gamma, G)$. *More precisely, if* $\rho \in \mathrm{Hom}_{\tau_{mod}}(\Gamma, G)$ *is* M*-Morse with respect to a base-point* $x \in X$ *then there exists a neighborhood of* ρ *in* $\mathrm{Hom}(\Gamma, G)$ *consisting entirely of* M'*-Morse representations with respect to* x*, where* M' *depends only on* M.

Proof. Let $\rho : \Gamma \curvearrowright X$ be a Morse action. We fix a word metric on Γ and a base point $x \in X$. Then there exist data $M = (L, A, \Theta, D)$ such that the orbit map $\Gamma \to \Gamma x \subset X$ extends to a (Θ, D, L, A)-Morse map of the Cayley graph Y on Γ.

We relax the Morse parameters slightly, i.e. we consider (L, A, Θ, D)-Morse quasigeodesics as $(L, A + 1, \Theta, D + 1)$-Morse quasigeodesics satisfying strict inequalities. For every scale S, the orbit map $\Gamma \to \Gamma x \subset X$, defines an $(L, A + 1, \Theta, D + 1, S)$-*local* Morse embedding $Y \to X$. Due to Γ-equivariance, this is a finite condition in the sense that it is equivalent to a condition involving only finitely many orbit points. Since we relaxed the Morse parameters, the same condition is satisfied by all actions sufficiently close to ρ.

Theorem 3.37 provides a scale S such that all S-local $(\Theta, D+1, L, A+1)$-Morse embeddings $Y \to X$ are M'-Morse for some Morse datum M' depending only on $(L, A + 1, \Theta, D + 1, S)$. It follows that actions sufficiently close to ρ are τ_{mod}-Morse. $\square$

Corollary 4.5. *For every hyperbolic group* Γ *the space of* faithful *Morse representations*

$$\mathrm{Hom}_{inj,\tau_{mod}}(\Gamma, G)$$

is open in $\mathrm{Hom}_{\tau_{mod}}(\Gamma, G)$.

Proof. Every hyperbolic group Γ has the unique maximal finite normal subgroup $\Phi \lhd \Gamma$ (if Γ is nonelementary then Φ is the kernel of the action of Γ on $\partial_\infty \Gamma$). Since Morse actions are properly discontinuous, the kernel of every Morse representation $\Gamma \to G$ is contained in Φ. Since $\mathrm{Hom}(\Phi, G)/G$ is finite, it follows that the set of faithful Morse representations is open in $\mathrm{Hom}_{\tau_{mod}}(\Gamma, G)$. $\square$

The result on the openness of the Morse condition for actions of word hyperbolic groups, cf. Theorem 4.4, can be strengthened in the sense that the asymptotics of Morse actions vary continuously:

Theorem 4.6 (Morse actions are structurally stable). *The boundary map at infinity of a Morse action depends continuously on the action.*

Proof. According to Theorem 4.4 nearby actions are uniformly Morse. The assertion therefore follows from the fact that the ends of Morse quasirays vary uniformly continuously, cf. Lemma 3.13. $\square$

Remark 4.7. (i) Note that since the boundary maps at infinity are embeddings, the Γ-actions on the τ_{mod}-limit sets are topologically conjugate to each other and, for nearby actions, by a homeomorphism close to the identity.

(ii) In rank one, our argument yields a different proof for Sullivan's Structural Stability Theorem [Su] for convex cocompact group actions on rank one symmetric spaces. Other proofs can be found in [La, GW] (for Anosov subgroups in higher rank), [Co, Iz, Bo] for rank one symmetric spaces.

Our next goal is to extend the topological conjugation from the limit set to the domains of proper discontinuity. Recall that in [KLP4] we constructed domains of proper discontinuity and cocompactness for τ_{mod}-Morse group actions on flag-manifolds $\mathrm{Flag}(\nu_{mod}) = G/P_{\nu_{mod}}$. Such domains depend on a certain auxiliary datum, a *balanced thickening* $\mathrm{Th} \subset W$, which is a $W_{\tau_{mod}}$-left invariant subset satisfying certain conditions; see [KLP4, sect. 3.4]. Let $\nu_{mod} \subset \sigma_{mod}$ be an ι-invariant face such that Th is invariant under the action of $W_{\nu_{mod}}$ via the *right* multiplication (this is automatic if $\nu_{mod} = \sigma_{mod}$ since $W_{\sigma_{mod}} = \{e\}$). The thickening $\mathrm{Th} \subset W$ defines a thickening $\mathrm{Th}(\Lambda_{\tau_{mod}}(\Gamma)) \subset \mathrm{Flag}(\nu_{mod})$. One of the main results of [KLP4] (Theorem 1.7) is that each τ_{mod}-Morse subgroup $\Gamma < G$ acts properly discontinuously and cocompactly on

$$\Omega_{\mathrm{Th}}(\Gamma) := \mathrm{Flag}(\nu_{mod}) - \mathrm{Th}(\Lambda_{\tau_{mod}}(\Gamma)).$$

Theorem 4.8 (Stability of Morse quotient spaces). *Suppose that $\rho_n : \Gamma \to \rho_n(\Gamma) = \Gamma_n < G$ is a sequence of faithful τ_{mod}-Morse representations converging to a τ_{mod}-Morse embedding $\rho : \Gamma \hookrightarrow G$. Then:*

1. *The sequence of thickenings $Th(\Lambda_{\tau_{mod}}(\Gamma_n))$ Hausdorff-converges to $Th(\Lambda_{\tau_{mod}}(\Gamma))$.*

2. *If $\gamma_n \in \Gamma$ is a divergent sequence, then, after extraction, the sequence $(\rho_n(\gamma_n))$ flag-converges to a point in $\Lambda_{\tau_{mod}}(\Gamma)$.*

3. There is a sequence of equivariant diffeomorphisms $h_n : \Omega_{Th}(\Gamma) \to \Omega_{Th}(\Gamma_n)$ converging to the identity map uniformly on compacts.

4. In particular, the quotient-orbifolds $\Omega_{Th}(\Gamma_n)/\Gamma_n$ are diffeomorphic to $\Omega_{Th}(\Gamma)/\Gamma$ for all sufficiently large n.

Proof. 1. First of all, suppose that a sequence $\tau_n \in \mathrm{Flag}(\tau_{mod})$ converges to $\tau \in \mathrm{Flag}(\tau_{mod})$. Then, since $\mathrm{Flag}(\nu_{mod}) = G/P_{\nu_{mod}}$, there is a sequence $g_n \in G$, $g_n \to e$, such that $g_n(\tau) = \tau_n$. Since

$$g_n(\mathrm{Th}(\tau)) = \mathrm{Th}(g_n\tau) = \mathrm{Th}(\tau_n),$$

it follows that we have Hausdorff-convergence of subsets $\mathrm{Th}(\tau_n) \to \mathrm{Th}(\tau)$. Moreover, this convergence of subsets is uniform: There exists $n_0 = n(\delta)$ such that if $d(\tau_n, \tau) < \delta$ for all $n \geqslant n_0$ then $d(\mathrm{Th}(\tau_n), \mathrm{Th}(\tau)) < \epsilon = \epsilon(\delta)$ for all $n \geqslant n_0$. Here $\epsilon \to 0$ as $\delta \to 0$. Since the sequence of limit sets $\Lambda_{\tau_{mod}}(\Gamma_n)$ Hausdorff-converges to $\Lambda_{\tau_{mod}}(\Gamma)$, it follows that the sequence of thickenings $\mathrm{Th}(\Lambda_{\tau_{mod}}(\Gamma_n))$ Hausdorff-converges to $\mathrm{Th}(\Lambda_{\tau_{mod}}(\Gamma))$. This proves (1).

2. Consider a sequence of geodesic rays $e\xi_n$ in the Cayley graph Y of Γ such that γ_n lies in an R-neighborhood of $e\xi_n$ for all n. Then, in view of the uniform M'-Morse property for the representations ρ_n, each point $\rho_n(\gamma_n)(x)$ belongs to the D'-neighborhood of the Weyl cone $V(x, \mathrm{st}(\tau_n))$, where $\tau_n = \alpha_n(\xi_n)$, $\alpha_n : \partial_\infty\Gamma \to \Lambda_{\tau_{mod}}(\Gamma_n)$ is the asymptotic embedding. Thus, by the definition of flag-convergence, the sequences $(\rho_n(\gamma_n))$ and (τ_n) have the same flag-limit in $\mathrm{Flag}(\tau_{mod})$. By Part 1, the sequence (τ_n) subconverges to a point in $\Lambda_{\tau_{mod}}(\Gamma)$. Hence, the same holds for $(\rho_n(\gamma_n))$.

3. The proof of this part is mostly standard, see [Iz] in the case when X is a hyperbolic space. The quotient orbifold $O = \Omega_{\mathrm{Th}}(\Gamma)/\Gamma$ has a natural (F, G)-structure where $\mathrm{F} = \mathrm{Flag}(\nu_{mod})$. The orbifold O has finitely many components, let Z be one of them and let $\hat{Z} \subset \Omega_{\mathrm{Th}}(\Gamma)$ be a component projecting to Z. It suffices to construct maps h_n on each component $\hat{Z}$ and then extend these maps to maps h_n of $\Omega_{\mathrm{Th}}(\Gamma)$ by ρ_n-equivariance.

The covering map $\hat{Z} \to Z$ induces an epimorphism $\phi : \pi_1(Z) \to \Gamma_Z$, where Γ_Z is the Γ-stabilizer of $\hat{Z}$. Let $dev : \tilde{Z} \to \hat{Z} \subset \Omega_{\mathrm{Th}}(\Gamma)$ be the developing map, where $\tilde{Z} \to Z$ is the universal covering. By Ehresmann-Thurston holonomy theorem (see [Lo], [CEG], [Go], [K1, sect. 7.1]), for all sufficiently large n, the homomorphism $\phi_n := \rho_n \circ \phi$ is the holonomy of an (F, G)-structure on Z. Moreover, the developing maps $dev_n : \tilde{Z} \to \mathrm{F}$ converge to dev uniformly on compacts in the C^∞-topology. Since $\pi_1(\hat{Z})$ is contained in the kernel of ϕ, it is also in the kernel of ϕ_n. Hence, the maps dev_n descend to maps $\widehat{dev_n} : \hat{Z} \to \mathrm{F}$. The sequence $\widehat{dev_n}$ still converges to the identity embedding $\hat{Z} \hookrightarrow \mathrm{F}$ uniformly on compacts. Pick a compact fundamental set $C \subset \hat{Z}$ for the Γ_Z-action, i.e. a compact subset whose Γ-orbit equals $\hat{Z}$. In view of Part 1 of the theorem, $\widehat{dev_n}(C) \subset \Omega_{\mathrm{Th}}(\Gamma_n)$ for all sufficiently large n. Therefore, we can assume that $\widehat{dev_n}(\hat{Z})$ is contained in a component $\hat{Z}_n$ of $\Omega_{\mathrm{Th}}(\Gamma_n)$. By the compactness of the quotient-orbifolds, $\widehat{dev_n}$ projects to a finite-to-one (smooth) orbi-covering map $c_n : Z \to Z_n := \hat{Z}_n/\rho_n(\Gamma_Z)$. Hence, $\widehat{dev_n} : \hat{Z} \to \hat{Z}_n$ is a covering map as well. If $\hat{Z}_n$ were simply-connected, it would follow that $\widehat{dev_n}$ is a diffeomorphism as required (and this is how Izeki concludes his proof in [Iz]). We will prove that $\widehat{dev}_n$ is a diffeomorphism by a direct argument.

Suppose that each $\widehat{dev}_n$ is not injective. Then, by the equivariance of these maps, after extraction, there exist convergent sequences $z_n \to z, z'_n \to z'$ in $\hat{Z}$ and a sequence $\gamma_n \in \Gamma$ such that

$$\rho_n(\gamma_n)\widehat{dev}_n(z_n) = \widehat{dev}_n(z'_n), \quad \gamma_n(z_n) \neq z'_n.$$

If the sequence (γ_n) were contained in a finite subset of Γ we would obtain a contradiction with the uniform convergence on compacts $\widehat{dev}_n \to id$ on $\hat{Z}$. Hence, after extraction, we may assume that (γ_n) is a divergent sequence. We, therefore, obtain a dynamical relation between the points z, z' via the sequence $(\rho_n(\gamma_n))$. According to Part 2, the sequence $(\rho_n(\gamma_n))$ flag-accumulates to $\Lambda_{\tau_{mod}}(\Gamma)$. The dynamical relation then contradicts fatness of the balanced thickening Th, see [KLP4, sect. 5.2] and the proof of Theorem 6.8 in [KLP4].

We conclude that the maps

$$\widehat{dev}_n : \hat{Z} \to \hat{Z}_n$$

are diffeomorphisms for all sufficiently large n. Since $\rho_n : \Gamma \to \Gamma_n$ are isomorphisms, equivariance of the developing maps implies that the maps $h_n : \Omega_{\mathrm{Th}}(\Gamma) \to \Omega_{\mathrm{Th}}(\Gamma_n)$ are diffeomorphisms for sufficiently large n.

4. This part is an immediate corollary of Part 3. $\square$

Remark 4.9. (i) In the case when X is a hyperbolic space, the equivariant diffeomorphism $h_n : \Omega(\Gamma) \to \Omega(\Gamma_n)$ combined with the equivariant homeomorphism of the limit sets $\Lambda(\Gamma) \to \Lambda(\Gamma_n)$ yield an equivariant homeomorphism $\partial_\infty X \to \partial_\infty X$, see [Tu, Iz]. Such an extension does not exist in higher rank since, in general, there is no equivariant homeomorphism of thickened limit sets $\mathrm{Th}(\Lambda_{\tau_{mod}}(\Gamma)) \to \mathrm{Th}(\Lambda_{\tau_{mod}}(\Gamma_n))$. This can be already seen for group actions on products of hyperbolic planes.

(ii) An analogue of Theorem 4.8 holds when we replace the group actions on flag-manifolds with actions on Finsler compactifications of the symmetric space and replace flag-manifold thickenings $\mathrm{Th}(\Lambda_{\tau_{mod}})$ with Finsler thickenings $\mathrm{Th}_{F\ddot{u}}(\Lambda_{\tau_{mod}}) \subset \partial_{F\ddot{u}} X$. Proving this requires extending Ehresmann–Thurston holonomy theorem to the category of smooth manifolds with corners and we will not pursue it here.

4.2 Schottky actions

In this section we apply our local-to-global result for straight sequences (Theorem 3.18) to construct Morse actions of free groups, generalizing and sharpening[1] Tits's ping-pong construction.

We consider two oriented τ_{mod}-regular geodesic lines a, b in X. Let $\tau_{\pm a}, \tau_{\pm b} \in \mathrm{Flag}(\tau_{mod})$ denote the simplices which they are τ-asymptotic to, and let $\theta_{\pm a}, \theta_{\pm b} \in \sigma_{mod}$ denote the types of their forward/backward ideal endpoints in $\partial_\infty X$. (Note that $\theta_{-a} = \iota(\theta_a)$ and $\theta_{-b} = \iota(\theta_b)$.) Let Θ be a compact convex subset of $\mathrm{ost}(\tau_{mod}) \subset \sigma_{mod}$, which is invariant under ι.

[1] In the sense that we obtain free subgroups which are not only embedded, but also asymptotically embedded in G.

Definition 4.10 (Generic pair of geodesics). We call the pair of geodesics (a, b) *generic* if the four simplices $\tau_{\pm a}, \tau_{\pm b}$ are pairwise opposite.

Let $\alpha, \beta \in G$ be axial isometries with axes a and b respectively and translating in the positive direction along these geodesics. Then $\tau_{\pm a}$ and $\tau_{\pm b}$ are the attractive/repulsive fixed points of α and β on $\mathrm{Flag}(\tau_{mod})$.

For every pair of numbers $m, n \in \mathbb{N}$ we consider the representation of the free group in two generators

$$\rho_{m,n} : F_2 = \langle A, B \rangle \to G$$

sending the generator A to α^m and B to β^n. We regard it as an isometric action $\rho_{m,n} : F_2 \curvearrowright X$.

Definition 4.11 (Schottky subgroup). A τ_{mod}*-Schottky subgroup* of G is a free τ_{mod}-asymptotically embedded subgroup of G.

If G has rank one, this definition amounts to the requirement that Γ is convex cocompact and free. Equivalently, this is a discrete finitely generated subgroup of G which contains no nontrivial elliptic and parabolic elements and has totally disconnected limit set (see see [K1]). We note that this definition essentially agrees with the standard definition of Schottky groups in rank 1 Lie groups, provided one allows fundamental domains at infinity for such groups to be bounded by pairwise disjoint compact submanifolds which need not be topological spheres, see [K1] for the detailed discussion.

Theorem 4.12 (Morse Schottky actions). *If the pair of geodesics (a, b) is generic and if $\theta_{\pm a}, \theta_{\pm b} \in \mathrm{int}(\Theta)$, then the action $\rho_{m,n}$ is Θ-Morse for sufficiently large m, n. Thus, such $\rho_{m,n}$ is injective and its image is a τ_{mod}-Schottky subgroup of G.*

Remark 4.13. In particular, these actions are faithful and undistorted, compare Remark 4.2.

Proof. Let $S = \{A^{\pm 1}, B^{\pm 1}\}$ be the standard generating set. We consider the sequences (γ_k) in F_2 with the property that $\gamma_k^{-1}\gamma_{k+1} \in S$ and $\gamma_{k+1} \neq \gamma_{k-1}$ for all k. They correspond to the geodesic segments in the Cayley tree of F_2 associated to S which connect vertices.

Let $x \in X$ be a base point. In view of Lemma 3.8 we must show that the corresponding sequences $(\gamma_k x)$ in the orbit $F_2 \cdot x$ are uniformly Θ-Morse. (Meaning e.g. that the maps $\mathbb{R} \to X$ sending the intervals $[k, k+1)$ to the points $\gamma_k x$ are uniform Θ-Morse quasigeodesics.) As in the proof of Theorem 3.34 we will obtain this by applying our local to global result for straight spaced sequences (Theorem 3.18) to the associated midpoint sequences. Note that the sequences $(\gamma_k x)$ themselves cannot expected to be straight.

Taking into account the Γ-action, the uniform straightness of all midpoint sequences depends on the geometry of a finite configuration in the orbit. It is a consequence of the following fact. Consider the midpoints $y_{\pm m}$ of the segments $x\alpha^{\pm m}(x)$ and $z_{\pm n}$ of the segments $x\beta^{\pm n}(x)$.

Lemma 4.14. *For sufficiently large m, n the quadruple $\{y_{\pm m}, z_{\pm n}\}$ is arbitrarily separated and Θ-regular. Moreover, for any of the four points, the segments connecting it to the other three points have arbitrarily small ζ-angles with the segment connecting it to x.*

Proof. The four points are arbitrarily separated from each other and from x because the axes a and b diverge from each other due to our genericity assumption.

By symmetry, it suffices to verify the rest of the assertion for the point y_m, i.e. we show that the segments $y_m y_{-m}$ and $y_m z_n$ are Θ-regular for large m, n and that $\lim_{m\to\infty} \angle^{\zeta}_{y_m}(x, y_{-m}) = 0$ and $\lim_{n,m\to\infty} \angle^{\zeta}_{y_m}(x, z_n) = 0$.

The orbit points $\alpha^{\pm m}x$ and the midpoints $y_{\pm m}$ are contained in a tubular neighborhood of the axis a. Therefore, the segments $y_m x$ and $y_m y_{-m}$ are Θ-regular for large m and $\angle_{y_m}(x, y_{-m}) \to 0$. This implies that also $\angle^{\zeta}_{y_m}(x, y_{-m}) \to 0$.

To verify the assertion for (y_m, z_n) we use that, due to genericity, the simplices τ_a and τ_b are opposite and we consider the parallel set $P = P(\tau_a, \tau_b)$. Since the geodesics a and b are forward asymptotic to P, it follows that the points x, y_m, z_n have uniformly bounded distance from P. We denote their projections to P by $\bar{x}, \bar{y}_m, \bar{z}_n$.

Let $\Theta'' \subset \mathrm{int}(\Theta)$ be an auxiliary Weyl convex subset such that $\theta_{\pm a}, \theta_{\pm b} \in \mathrm{int}(\Theta'')$. We have that $\bar{y}_m \in V(\bar{x}, \mathrm{st}_{\Theta''}(\tau_a))$ for large m because the points y_m lie in a tubular neighborhood of the ray with initial point $\bar{x}$ and asymptotic to a. Similarly, $\bar{z}_n \in V(\bar{x}, \mathrm{st}_{\Theta''}(\tau_b))$ for large n. It follows that $\bar{x} \in V(\bar{y}_m, \mathrm{st}_{\Theta''}(\tau_b))$ and, using the convexity of Θ-cones (Proposition 2.1), that $\bar{z}_n \in V(\bar{y}_m, \mathrm{st}_{\Theta''}(\tau_b))$.

The cone $V(y_m, \mathrm{st}_{\Theta''}(\tau_b))$ is uniformly Hausdorff close to the cone $V(\bar{y}_m, \mathrm{st}_{\Theta''}(\tau_b))$ because the Hausdorff distance of the cones is bounded by the distance $d(y_m, \bar{y}_m)$ of their tips. Hence there exist points $x', z'_n \in V(y_m, \mathrm{st}_{\Theta''}(\tau_b))$ uniformly close to x, z_n. Since $d(y_m, x'), d(y_m, z'_n) \to \infty$ as $m, n \to \infty$, it follows that the segments $y_m x$ and $y_m z_n$ are Θ-regular for large m, n. Furthermore, since $\angle^{\zeta}_{y_m}(x', z'_n) = 0$ and $\angle_{y_m}(x, x') \to 0$ as well as $\angle_{y_m}(z_n, z'_n) \to 0$, it follows that $\angle^{\zeta}_{y_m}(x, z_n) \to 0$. ☐

Proof of Theorem concluded. The lemma implies that for any given l, ϵ the midpoint triples of the four point sequences $(\gamma_k x)$ are (Θ, ϵ)-straight and l-spaced if m, n are sufficiently large, compare the quadruple condition (Definition 3.31). This means that the midpoint sequences of all sequences $(\gamma_k x)$ are (Θ, ϵ)-straight and l-spaced for large m, n. Theorem 3.18 then implies that the sequences $(\gamma_k x)$ are uniformly Θ-Morse. ☐

Remark 4.15. 1. Generalizing the above argument to free groups with finitely many generators, one can construct Morse Schottky subgroups for which the set $\theta(\Lambda) \subset \sigma_{mod}$ of types of limit points is arbitrarily Hausdorff close to a given ι-invariant Weyl convex subset Θ. This provides an alternative approach to the second main theorem in [Be] using coarse geometric arguments.

2. In [DKL] Theorem 4.12 was generalized (by arguments similar to the its proof) to free products of Morse subgroups of G.

4.3 Algorithmic recognition of Morse actions

In this section, we describe an algorithm which has an isometric action $\rho : \Gamma \curvearrowright X$ and a point $x \in X$ as its input and terminates if and only if the action ρ is Morse (otherwise, the algorithm runs forever).

We begin by describing briefly the *Riley's algorithm* (see [Ri]) accomplishing a similar task, namely, detecting geometrically finite actions on $X = \mathbb{H}^3$. Suppose that we are given a finite (symmetric) set of generators $g_1 = 1, \ldots, g_m$ of a subgroup $\Gamma \subset PO(3,1)$ and a base-point $x \in X = \mathbb{H}^3$. The idea of the algorithm is to construct a finite sided Dirichlet fundamental domain D for Γ (with the center at x): Every geometrically finite subgroup of $PO(3,1)$ admits such a domain. (The latter is false for geometrically finite subgroups of $PO(n,1)$, $n \geqslant 4$, but is, nevertheless, true for convex cocompact subgroups.) Given a finite sided convex fundamental domain, one concludes that Γ is geometrically finite. Here is how the algorithm works: For each k define the subset $S_k \subset \Gamma$ represented by words of length $\leqslant k$ in the letters $g_1, \ldots, g_m$. For each $g \in S_k$ consider the half-space $Bis(x, g(x)) \subset X$ bounded by the bisector of the segment $xg(x)$ and containing the point x. Then compute the intersection

$$D_k = \bigcap_{g \in S_k} Bis(x, g(x)).$$

Check if D_k satisfies the conditions of the *Poincaré's Fundamental Domain theorem*. If it does, then $D = D_k$ is a finite sided fundamental domain of Γ. If not, increase k by 1 and repeat the process. Clearly, this process terminates if and only if Γ is geometrically finite.

One can enhance the algorithm in order to detect if a geometrically finite group is convex cocompact. Namely, after a Dirichlet domain D is constructed, one checks for the following:

1. If the ideal boundary of a Dirichlet domain D has isolated ideal points (they would correspond to rank two cusps which are not allowed in convex cocompact groups).

2. If the ideal boundary of D contains tangent circular arcs with points of tangency fixed by parabolic elements (coming from the "ideal vertex cycles"). Such points correspond to rank 1 cusps, which again are not allowed in convex cocompact groups.

Checking 1 and 2 is a finite process; after its completion, one concludes that Γ is convex cocompact.

We refer the reader to [Gi1, Gi2, GiM, K2] and [KL2, sect. 1.8] for more details concerning discreteness algorithms for groups acting on hyperbolic planes and hyperbolic 3-spaces.

We now consider group actions on general symmetric spaces. Let Γ be a hyperbolic group with a fixed finite (symmetric) generating set; we equip the group Γ with the word metric determined by this generating set.

For each n, let $\mathcal{L}_n$ denote the set of maps $q : [0, 3n] \cap \mathbb{Z} \to \Gamma$ which are restrictions of geodesics $\tilde{q} : \mathbb{Z} \to \Gamma$, such that $q(0) = 1 \in \Gamma$. In view of the geodesic automatic structure on Γ (see e.g. [Ep, Theorem 3.4.5]), the set $\mathcal{L}_n$ can be described via a finite state automaton.

Suppose that $\rho : \Gamma \curvearrowright X$ is an isometric action on a symmetric space X; we fix a base-point $x \in X$ and the corresponding orbit map $f : \Gamma \to \Gamma x \subset X$. We also fix an ι-invariant face τ_{mod} of the model spherical simplex σ_{mod} of X. The algorithm that we are about to describe will detect that the action ρ is τ_{mod}-Morse.

Remark 4.16. If the face τ_{mod} is not fixed in advance, we would run algorithms for each face τ_{mod} in parallel.

For the algorithm we will be using a special (countable) increasing family of Weyl convex compact subsets $\Theta = \Theta_i \subset \text{ost}(\tau_{mod}) \subset \sigma_{mod}$ which exhausts $\text{ost}(\tau_{mod})$; in particular, every compact ι-invariant convex subset of $\text{ost}(\tau_{mod}) \subset \sigma_{mod}$ is contained in some Θ_i:

$$\Theta_i := \{v \in \sigma : \min_{\alpha \in \Phi_{\tau_{mod}}} \alpha(v) \geqslant \frac{1}{i}\}, \tag{4.17}$$

where $\Phi_{\tau_{mod}}$ is the subset of the set of simple roots Φ (with respect to σ_{mod}) which vanish on the face τ_{mod}. Clearly, the sets Θ_i satisfy the required properties. Furthermore, we consider only those L and D which are natural numbers.

Next, consider the sequence

$$(L_i, \Theta_i, D_i) = (i, \Theta_i, D_i), i \in \mathbb{N}.$$

In order to detect τ_{mod}-Morse actions we will use the local characterization of Morse quasi-geodesics given by Theorem 3.18 and Proposition 3.32. Due to the discrete nature of quasi-geodesics that we will be considering, it suffices to assume that the additive quasi-isometry constant A is zero.

Consider the functions

$$l(\Theta, \Theta', \delta), \epsilon(\Theta, \Theta', \delta)$$

as in Theorem 3.18. Using these functions, for the sets $\Theta = \Theta_i, \Theta' = \Theta_{i+1}$ and the constant $\delta = 1$ we define the numbers

$$l_i = l(\Theta, \Theta', \delta), \epsilon_i = \epsilon(\Theta, \Theta', \delta).$$

Next, for the numbers $L = L_i, D = D_i$ and the sets $\Theta = \Theta_i, \Theta' = \Theta_{i+1}$, consider the numbers

$$s_i = s(L_i, 0, \Theta_i, \Theta_{i+1}, D_i, \epsilon_{i+1}, l_{i+1})$$

as in Proposition 3.32. According to this proposition, every $(L_i, 0, \Theta_i, D_i)$-Morse quasigeodesic satisfies the $(\Theta_{i+1}, \epsilon_{i+1}, l_{i+1}, s)$-quadruple condition for all $s \geqslant s_i$. We note that, a priori, the sequence s_i need not be increasing. We set $S_1 = s_1$ and define a monotonic sequence S_i recursively by

$$S_{i+1} = \max(S_i, s_{i+1}).$$

Then every $(\Theta_i, D_i, L_i, 0)$-Morse quasigeodesic also satisfies the $(\Theta_{i+1}, \epsilon_{i+1}, l_{i+1}, S_{i+1})$-quadruple condition.

We are now ready to describe the algorithm. For each $i \in \mathbb{N}$ we compute the numbers l_i, ϵ_i and, then, S_i, as above. We then consider finite discrete paths in Γ, $q \in \mathcal{L}_{S_i}$, and the corresponding discrete paths in X, $p(t) = q(t)x$, $t \in [0, 3S_i] \cap \mathbb{Z}$. The number of paths q (and, hence, p) for each i is finite, bounded by the growth function of the group Γ.

For each discrete path p we check the $(\Theta_i, \epsilon_i, l_i, S_i)$-quadruple condition. If for some $i = i_*$, all paths p satisfy this condition, the algorithm terminates: It follows from Theorem 3.18 that the map f sends all normalized discrete biinfinite geodesics in Γ to Morse quasigeodesics in X. Hence, the action $\Gamma \curvearrowright X$ is Morse in this case. Conversely, suppose that the action of Γ is $(\Theta, D, L, 0)$-Morse. Then f sends all isomeric embeddings $\tilde{q} : \mathbb{Z} \to \Gamma$ to $(\Theta, D, L, 0)$-Morse quasigeodesics $\tilde{p}$ in X. In view of the properties of the sequence

$$(L_i, \Theta_i, D_i),$$

it follows that for some i,

$$(L, \Theta, D) \leqslant (L_i, \Theta_i, D_i),$$

i.e., $L \leqslant L_i, \Theta \subset \Theta_i, D \leqslant D_i$; hence, all the biinfinite discrete paths $\tilde{p}$ are $(\Theta_i, D_i, L_i, 0)$-Morse quasigeodesic. By the definition of the numbers l_i, ϵ_i, S_i, it then follows that all the discrete paths $p = f \circ q, q \in \mathcal{L}_{S_i}$ satisfy the $(\Theta_{i+1}, \epsilon_{i+1}, l_{i+1}, S_{i+1})$-quadruple condition. Thus, the algorithm will terminate at the step $i + 1$ in this case.

Therefore, the algorithm terminates if and only if the action is Morse (for some parameters). If the action is not Morse, the algorithm will run forever. ☐

Remark 4.18. Applied to a rank one symmetric space X and a hyperbolic group Γ without a nontrivial normal finite subgroup, the above algorithm verifies if the given representation $\rho : \Gamma \to \mathrm{Isom}(X)$ is faithful with convex-cocompact image. We could not find this result in the existing literature; cf. however [GK].

5 Appendix: Further properties of Morse quasigeodesics

This is the only part of the paper not contained in [KLP1]. Here we collect various properties of Morse quasigeodesics that we found to be useful elsewhere in our work.

5.1 Finsler geometry of symmetric spaces

In [KL1], see also [KLP5], we considered a certain class of G-invariant "polyhedral" Finsler metrics on X. Their geometric and asymptotic properties turned out to be well adapted to the study of geometric and dynamical properties of regular subgroups. They provide a Finsler geodesic *combing* of X which is, in many ways, more suitable for analyzing the asymptotic geometry of X than the geodesic combing given by the standard Riemannian metric on X. These Finsler metrics also play a basic role in the present paper. We briefly recall their definition and some basic properties, and refer to [KL1, §5.1] for more details.

Let $\bar{\theta} \in \mathrm{int}(\tau_{mod})$ be a type spanning the face type τ_{mod}. The $\bar{\theta}$-*Finsler distance* $d^{\bar{\theta}}$ on X is the G-invariant pseudo-metric defined by

$$d^{\bar{\theta}}(x, y) := \max_{\theta(\xi)=\bar{\theta}} \big(b_\xi(x) - b_\xi(y)\big)$$

for $x, y \in X$, where the maximum is taken over all ideal points $\xi \in \partial_\infty X$ with type $\theta(\xi) = \bar{\theta}$. It is positive, i.e. a (non-symmetric) metric, if and only if the radius of σ_{mod} with respect to $\bar{\theta}$ is $< \frac{\pi}{2}$. This is in turn equivalent to $\bar{\theta}$ not being contained in a factor of a nontrivial spherical join decomposition of σ_{mod}, and is always satisfied e.g. if X is irreducible.

If $d^{\bar{\theta}}$ is positive, it is equivalent to the Riemannian metric. In general, if it is only a pseudo-metric, it is still equivalent to the Riemannian metric d on uniformly regular pairs of points. More precisely, if the pair of points x, y is Θ-regular, then

$$L^{-1}d(x,y) \leqslant d^{\bar{\theta}}(x,y) \leqslant Ld(x,y)$$

with a constant $L = L(\Theta) \geqslant 1$.

Regarding symmetry of the Finsler distance, one has the identity

$$d^{\iota\bar{\theta}}(y,x) = d^{\bar{\theta}}(x,y)$$

and hence $d^{\bar{\theta}}$ is symmetric if and only if $\iota\bar{\theta} = \bar{\theta}$. We refer to $d^{\bar{\theta}}$ as a Finsler metric *of type* τ_{mod}.

The $d^{\bar{\theta}}$-balls in X are convex but not strictly convex. (Their intersections with flats through their centers are polyhedra.) Accordingly, $d^{\bar{\theta}}$-geodesics connecting two given points x, y are not unique. To simplify notation, xy will stand for *some* $d^{\bar{\theta}}$-geodesic connecting x and y. The union of all $d^{\bar{\theta}}$-geodesic xy equals the τ_{mod}-diamond $\diamondsuit_{\tau_{mod}}(x,y)$, that is, a point lies on a $d^{\bar{\theta}}$-geodesic xy if and only if it is contained in $\diamondsuit_{\tau_{mod}}(x,y)$, see [KLP5]. Finsler geometry thus provides an alternative description of diamonds. Note that with this description, the diamond $\diamondsuit_{\tau_{mod}}(x,y)$ is also defined when the segment xy is not τ_{mod}-regular. Such a *degenerate* τ_{mod}-diamond is contained in a smaller totally-geodesic subspace, namely in the intersection of all τ_{mod}-parallel sets containing the points x, y. The description of geodesics and diamonds also implies that the unparameterized $d^{\bar{\theta}}$-geodesics depend only on the face type τ_{mod}, and not on $\bar{\theta}$. We will refer to $d^{\bar{\theta}}$-geodesics as τ_{mod}-*Finsler geodesics*. Note that Riemannian geodesics are Finsler geodesics.

We will call a Θ-regular τ_{mod}-Finsler geodesic a Θ-*Finsler geodesic*. If xy is a Θ-regular (Riemannian) segment, then the union of Θ-Finsler geodesics xy equals the Θ-diamond $\diamondsuit_\Theta(x,y)$.

Every τ_{mod}-Finsler ray in X is contained in a τ_{mod}-Weyl cone, and we will use the notation $x\tau$ for a τ_{mod}-Finsler ray contained $V(x, \mathrm{st}(\tau))$. Similarly, every τ_{mod}-Finsler line is contained in a τ_{mod}-parallel set, and we denote by $\tau_-\tau_+$ an oriented τ_{mod}-Finsler line forward/backward asymptotic to two antipodal simplices $\tau_\pm \in \mathrm{Flag}(\tau_{mod})$ and contained in $P(\tau_-, \tau_+)$.

Examples of Θ-regular Finsler geodesics can be obtained as follows. Let (x_i) be a (finite or infinite) sequence contained in a parallel set $P(\tau_-, \tau_+)$ such that each Riemannian segment x_ix_{i+1} is τ_+-longitudinal and Θ'-regular. Then the concatenation of these geodesic segments is

Conversely, every Θ-regular Finsler geodesic $c : I \to X$ can be *approximated* by a piecewise-Riemannian Finsler geodesic c': Pick a number $s > 0$ and consider a maximal s-separated subset $J \subset I$. Then take c' to be the concatenation of Riemannian geodesic segments $c(i)c(j)$ for consecutive pairs $i, j \in J$. In view of this approximation procedure, the String of Diamonds Theorem (Theorem 3.30) holds if instead of Riemannian geodesic segments x_ix_{i+1} we allow Θ-regular Finsler segments.

5.2 Stability of diamonds

Diamonds can be regarded as Finsler-geometric replacements of *geodesic segments* in nonpositively curved symmetric spaces of higher rank.

Riemannian geodesic segments in Hadamard manifolds (and, more generally, $CAT(0)$ metric spaces) depend *uniformly continuously* on their tips: By convexity of the distance function we have,

$$d_{Haus}(xy, x'y') \leqslant \max(d(x,x'), d(y,y')).$$

In [KLP2, Prop. 3.70] we proved that diamonds $\diamondsuit_{\tau_{mod}}$ depend *continuously* on their tips.

Below we establish uniform control on how much sufficiently large Θ-diamonds vary with their tips.

Lemma 5.1. *For $d' > d > 0$ there exists $C = C(\Theta, \Theta', d, d')$ such that the following holds:*

If a segment $x_-x_+ \subset X$ is Θ-regular with length $\geqslant C$ and $y_\pm \in B(x_\pm, d)$, then the segment y_-y_+ is Θ'-regular and $\diamondsuit_\Theta(x_-, x_+) \subset N_{d'}(\diamondsuit_{\Theta'}(y_-, y_+))$.

Proof. The Θ'-regularity of y_-y_+ for sufficiently large C follows from the Δ-triangle inequality.

Suppose that there exists no constant C for which also the second assertion holds. Then there are sequences of points $x_n^\pm$ with $d(x_n^-, x_n^+) \to +\infty$, $y_n^\pm$ with $d(x_n^\pm, y_n^\pm) \leqslant d$, $x_n \in \diamondsuit_\Theta(x_n^-, x_n^+)$ and $y_n \in \diamondsuit_{\Theta'}(y_n^-, y_n^+)$ with $d(x_n, \diamondsuit_{\Theta'}(y_n^-, y_n^+)) = d(x_n, y_n) = d'$. We may assume convergence $x_n \to x_\infty$ and $y_n \to y_\infty$ in X.

After extraction, at least one of the sequences $(x_n^\pm)$ diverges. There are two cases to consider.

Suppose first that both sequences $(x_n^\pm)$ diverge. Then they are uniformly τ_{mod}-regular and, after extraction, we have τ_{mod}-flag convergence $x_n^\pm, y_n^\pm \to \tau_\pm \in \mathrm{Flag}(\tau_{mod})$. The limit simplices $\tau_\pm$ are antipodal (because $x_n \to x_\infty$). We observe that

$$d(x_n, \partial\diamondsuit_{\Theta'}(x_n^-, x_n^+)), d(y_n, \partial\diamondsuit_{\Theta'}(y_n^-, y_n^+)) \to +\infty.$$

It follows that the sequences of diamonds $\diamondsuit_{\Theta'}(x_n^-, x_n^+)$ and $\diamondsuit_{\Theta'}(y_n^-, y_n^+)$ both Hausdorff converge to the τ_{mod}-parallel set $P = P(\tau_-, \tau_+)$. It holds that $x_\infty \in P$ because $x_n \in \diamondsuit_\Theta(x_n^-, x_n^+)$. On the other hand, $d(x_\infty, P) = d'$ because $d(x_n, \diamondsuit_{\Theta'}(y_n^-, y_n^+)) = d'$, a contradiction.

Second, suppose that only one of the sequences $(x_n^\pm)$ diverges, say, after extraction, $x_n^- \to x_\infty^-$ and $y_n^- \to y_\infty^-$ in X to limit points with $d(x_\infty^-, y_\infty^-) \leqslant d$, and $x_n^+ \to \tau_+ \in \mathrm{Flag}(\tau_{mod})$. Now the distance of x_n from the boundary of the Θ'-Weyl cone with tip x_n^+ and containing x_n goes to infinity and it follows that $\diamondsuit_{\Theta'}(x_n^-, x_n^+) \to V(x_\infty^-, \mathrm{st}_{\Theta'}(\tau_+))$ and, similarly, $\diamondsuit_{\Theta'}(y_n^-, y_n^+) \to V(y_\infty^-, \mathrm{st}_{\Theta'}(\tau_+))$. The asymptotic limit Weyl cones have Hausdorff distance $d(x_\infty^-, y_\infty^-)$. On the other hand, $x_\infty \in V(x_\infty^-, \mathrm{st}_{\Theta'}(\tau_+))$ and $d(x_\infty, V(y_\infty^-, \mathrm{st}_{\Theta'}(\tau_+))) = d'$, again a contradiction.

This shows that also (ii) holds for sufficiently large C. $\square$

We reformulate this result in terms of Finsler geodesics:

Lemma 5.2. *There exists $C = C(\Theta, \Theta', d, d')$ such that the following holds: If x_-x_+ is a Θ-Finsler geodesic in X with $d(x_-, x_+) \geqslant C$ and $y_\pm$ are points with $d(y_\pm, x_\pm) \leqslant d$, then every* *point x on x_-x_+ lies within distance d' of a point y on a Θ'-Finsler geodesic y_-y_+.*

Note that we do not claim here that one can take the same Finsler geodesic y_-y_+ for all points x on x_-x_+.

We now apply this stabilty result to Morse quasigeodesics. One, somewhat annoying, feature of the definition of Θ-Morse quasigeodesics $p : I \to X$ is that $p([t_1, t_2])$ is not required to be uniformly close to a Θ-diamond spanned by $p(t_1), p(t_2)$. (One reason is because the segment $p(t_1)p(t_2)$ need not be Θ-regular.) Nevertheless, Lemma 5.1 implies:

Lemma 5.3. *For every Morse datum $M = (\Theta, B, L, A)$ and $\Theta' > \Theta$, there exists $C = C(M, \Theta')$ and D' such that whenever $d(x_1, x_2) \geqslant C$, the segment $x_1x_2 = p(t_1)p(t_2)$ is Θ'-regular and $p([t_1, t_2])$ lies in the D'-neighborhood of the Θ'-diamond $\diamondsuit_{\Theta'}(x_1, x_2)$.*

5.3 Finsler approximation of Morse quasigeodesics

The next theorem establishes that every (sufficiently long) Morse quasigeodesic is uniformly close to a Finsler geodesic with the same end-points. In this theorem, for convenience of the notation, we will be allowing Morse quasigeodesics p to be defined on closed intervals I in the extended real line; this is just a shorthand for a map $I' = I \cap \mathbb{R} \to X$ such that, as $I' \ni t \to \pm\infty$, $p(t) \to p(\pm\infty) \in \mathrm{Flag}(\tau_{mod})$. When we say that such maps p, c are within distance D' from each other, this simply means that their restrictions to I' are within distance $\leqslant D'$.

Theorem 5.4 (Finsler approximation theorem). *For every Morse datum $M = (\Theta, D, L, A)$, $\Theta' > \Theta$, and a positive number S, there exist $C = C(M, \Theta', S), D' = D'(M, \Theta', S)$ satisfying the following.*

Let $p : I = [t_-, t_+] \to X \cup \mathrm{Flag}(\tau_{mod})$ be a M-Morse quasigeodesic between the points $x_\pm = p(t_\pm) \in X \cup \mathrm{Flag}(\tau_{mod})$ such that $d(x_-, x_+) \geqslant C$. Then there exists a Θ'-Finsler geodesic x_-x_+ equipped with a monotonic parameterization $c : I \to x_-x_+$ such that:

(a) The maps $p, c : I \to X$ are within distance $\leqslant D'$ from each other.

(b) x_-x_+ is an S-spaced piecewise-Riemannian geodesic, i.e. the Riemannian length of each Riemannian segments of x_-x_+ is $\geqslant S$.

Proof. We will prove this in the case when both $x_\pm$ are in X since the proofs when one or both points $x_\pm$ are in $\mathrm{Flag}(\tau_{mod})$ are similar: One replaces diamonds with Weyl cones or parallel sets.

By the definition of an M-Morse quasigeodesic, for all subintervals $[s_-, s_+] \subset [t_-, t_+]$, there exists a Θ-diamond

$$\diamondsuit_\Theta(y'_-, y'_+)$$

whose D-neighborhood contains $p([s_-, s_+])$, and for $y_\pm = p(s_\pm)$, we have

$$d(y_\pm, y'_\pm) \leqslant D.$$

Therefore, applying the first part of Lemma 5.1, we conclude that the Riemannian segment y_-y_+ is Θ'-regular provided that $d(y_-, y_+) \geqslant C_1 = C_1(M, \Theta')$. In view of the quasigeodesic property of p, the last inequality follows from the separation condition

$$s_+ - s_- \geqslant s = s(M, \Theta').$$

This, of course, also applies to $[s_-, s_+] = [t_-, t_+]$ and, hence, using the second part of Lemma 5.1, we obtain

$$p(I) \subset N_D\left(\diamondsuit_\Theta(x'_-, x'_+)\right) \subset N_{D+D_1}\left(\diamondsuit_{\Theta'}(x_-, x_+)\right),$$

where $D_1 = D_1(M, \Theta')$. We let

$$\bar{y}_\pm \in \diamondsuit' := \diamondsuit_{\Theta'}(x_-, x_+) = V(x_-, \mathrm{st}_{\Theta'}(\tau_+)) \cap V(x_+, \mathrm{st}_{\Theta'}(\tau_-))$$

denote the nearest-point projections of $y_\pm = p(s_\pm)$. As long as $s_+ - s_- \geqslant s'(M, \Theta')$, the Riemannian segments $\bar{y}_-\bar{y}_+$ are also Θ'-regular and have length $\geqslant S$. Furthermore, as in the proof of Proposition 3.32, we can choose s' such that each segment $\bar{y}_-\bar{y}_+$ is τ_+-longitudinal.

We assume, from now on, that $t_+ - t_- \geqslant s''(M, \Theta')$, which is achieved by assuming that

$$L^{-1}(d(x_-, x_+) - A) \geqslant s'(M, \Theta').$$

Take a maximal s'-separated subset $J \subset I$ containing $t_\pm$. For each $j \in J$ define the point

$$z_j := \overline{p(j)} \in \diamondsuit'.$$

Then for all consecutive $i, j \in J$, $s' \leqslant |j - i| \leqslant 2s'$ we have

$$L^{-1}s' - (A + 2D + 2D_1) \leqslant d(z_i, z_j) \leqslant 2Ls' + (A + 2D + 2D_1). \tag{5.5}$$

We then let c denote the concatenation of Riemannian segments z_iz_j for consecutive $i, j \in J$, where we use the affine parameterization of $[i, j] \to z_iz_j$. Thus, c is a Θ'-Finsler geodesic. We now take the smallest $s'' \geqslant s'(M, \Theta')$ satisfying

$$S \leqslant L^{-1}s'' - (A + 2D + 2D_1),$$

the inequalities (5.5) imply that c satisfies both requirements of the approximation theorem with

$$D' = 2Ls'' + (A + 2D + 2D_1) + (D + D_1) + (2Ls'' + A). \quad \square$$

Remark 5.6. In the case when the domain of p is unbounded, one can prove a bit sharper result, namely, one can take $\Theta' = \Theta$. Compare [KL3, sect. 6].

5.4 Altering Morse quasigeodesics

Below we consider certain modifications of M-Morse quasigeodesics p in X represented as concatenations $p = p_- \star p_0 \star p_+$, where $x_\pm$ are the end-points of p_0, and $y_\pm, x_\pm$ are the end-points of $p_\pm$. (As in the previous section, we will be allowing $y_\pm$ to be in $X \cup \mathrm{Flag}(\tau_{mod})$.)

These modifications will have the form $p' = p'_- \star p'_0 \star p'_+$, where $p'_\pm$ and p'_0 are all Morse. We will see that, under certain assumptions, the entire p' is again Morse (for suitable Morse datum M').

We begin by analyzing extensions of p to biinfinite paths.

Lemma 5.7 (Extension lemma). *Suppose that*

$$p_\pm \subset V_\pm = V(x_\pm, \mathrm{st}(\tau_\pm)).$$

Whenever $y_\pm$ is in X, we let $c_\pm$ be Θ-regular Finsler rays contained in $V_\pm$ and connecting $y_\pm$ to $\tau_\pm$. Then, for every $\Theta' > \Theta$, there exists a Morse datum M' containing Θ' such that the concatenation

$$\hat{p} = c_- \star p \star c_+$$

is M'-Morse, provided that $d(x_\pm, y_\pm) \geqslant C = C(M, \Theta')$.

Proof. We fix an auxiliary subset Θ_1 satisfying $\Theta < \Theta_1 < \Theta'$. We let $S = S(\Theta_1, \Theta', 1), \epsilon = \epsilon(\Theta_1, \Theta', 1)$ be constants as in the string of diamonds theorem (Theorem 3.30).

According to Theorem 5.4, there exists a Θ'-regular Finsler geodesic

$$\bar{c} = y_-\bar{x}_- \star \bar{x}_-\bar{x}_+ \star \bar{x}_+y_+$$

within distance $D_1 = D_1(M, \Theta', S)$ from the path p, such that $\bar{c}$ is the concatenation of segments of length $\geqslant S$ and $d(x_\pm, \bar{x}_\pm) \leqslant D_1$. We let $z_\pm y_\pm$ denote the subsegments of $\bar{x}_\pm y_\pm$ containing $y_\pm$.

Since $d(x_\pm, \bar{x}_\pm) \leqslant D_1$, for each $\epsilon > 0$ and a sufficiently large $C_1 = C_1(D_1, \Theta')$, the inequality $d(x_\pm, y_\pm) \geqslant C_1$ implies

$$\angle^\zeta_{y_\pm}(x_\pm, \bar{x}_\pm) \leqslant \epsilon.$$

Therefore,

$$\angle^\zeta_{y_\pm}(z_\pm, \tau_\pm) \geqslant \pi - \epsilon$$

and, hence, the piecewise-geodesic path

$$\hat{c} = c_- \star \bar{c} \star c_+$$

is (Θ_1, ϵ)-straight and S-spaced. Hence, by Theorem 3.30, the concatenation $\hat{c}$ is M'-Morse, where $M_1 = (\Theta', 1, L, A)$. Since the path $\hat{p}$ is within distance D_1 from $\hat{c}$, it is M'-Morse, where $M' = M_1 + D_1$. ◻

The next lemma was proven in [DKL, Thm. 4.11] in the case when p, p' are finite paths. The proof in the case of (bi)infinite paths is the same and we omit it.

Lemma 5.8 (Replacement lemma). *Suppose that $p' = p'_- \star p'_0 \star p'_+$ is a concatenation of M-Morse quasigeodesics in X, such that the end-points of $p_\pm, p'_\pm$ and p_0, p'_0 are the same. Then for every $\Theta' > \Theta$ there exists a Morse datum M' containing Θ' such that the path p' is M'-Morse.*

In the following lemmata we will modify the path p by altering $p_\pm$ and keeping p_0 unchanged or moving it by a small amount ("wiggling the head and the tail of p").

Lemma 5.9 (Wiggle lemma, I). *Suppose that the paths $p_\pm, p'_\pm$ are both infinite. We let $p'_\pm$ be M-Morse quasigeodesics with finite terminal points $x_\pm$ and set $p' := p'_- \star p_0 \star p'_+$. Given $\Theta' > \Theta$ there exists $\epsilon = \epsilon(M, \Theta') > 0$ and a Morse datum M' containing Θ' such that if*

$$\mu := \max(\angle^{\zeta}_{x_\pm}(p'_\pm(\pm\infty), p_\pm(\pm\infty))) < \epsilon,$$

then p' is M'-Morse.

Proof. We fix an auxiliary compact Weyl-convex subset $\Theta_1 \subset \mathrm{ost}(\tau_{mod})$ such that $\Theta < \Theta_1 < \Theta'$. Set $\tau_\pm = p_\pm(\pm\infty)$, $\tau'_\pm = p'_\pm(\pm\infty)$.

According to Lemma 5.8, there exists a Morse datum M_1 containing Θ_1 such that for any Θ_1-regular Finsler geodesic rays $c_\pm := x_\pm\tau_\pm$, the concatenation $c_- \star p_0 \star c_+$ is M_1-Morse.

Let $M_2 > M_1 + 1$ be a Morse datum containing Θ' and let $S > 0$ be such that if a path q in X is S-locally $M_1 + 1$-Morse then q is M_2-Morse (see Theorem 3.34). Let ϵ be such that for $x \in X, \tau, \tau' \in \mathrm{Flag}(\tau_{mod})$, if $\angle^{\zeta}_x(\tau, \tau') < \epsilon$ then each Θ_1-regular Finsler segment of length $\leqslant S$ in $V(x, \mathrm{st}(\tau'))$ is within unit distance from a Θ_1-regular Finsler segment of length $\leqslant S$ in $V(x, \mathrm{st}(\tau))$. We assume now that $\mu < \epsilon$.

Since $p'_\pm$ are M-Morse rays, they are within distance $D_1 = D_1(M, \Theta_1)$ from Θ_1-regular Finsler rays $c'_\pm = x_\pm\tau'_\pm$ connecting $x_\pm$ and $\tau'_\pm$. Define a new path $c' := c'_- \star p_0 \star c'_+$.

By our choice of ϵ, the Θ_1-regular Finsler subsegment $s'_\pm = x_\pm y'_\pm$ of $c'_\pm$ of length S is within unit distance from a Θ_1-regular Finsler subsegment $s_\pm = x_\pm y_\pm$ of $c_\pm$ of length S, where $c_\pm = x_\pm\tau_\pm$ is a Θ_1-Finsler geodesic connecting $x_\pm$ to $\tau_\pm$.

The concatenation

$$s_- \star p_0 \star s_+$$

is M_1-Morse, and, since $c'_\pm$ are Θ_1-Finsler geodesic, the path c' is S-locally $M_1 + 1$-Morse. By our choice of S, the path c' is M_2-Morse. Since c' is within distance D_1 from p', the path p' is $M_2 + D_1$-Morse. Lastly, we set $M' := M_2 + D_1$. ☐

We generalize this lemma by allowing finite Morse quasigeodesics. We continue with the setting of Lemma 5.9; we now allow paths $p_\pm$ and $p'_\pm$ to be finite, connecting $y_\pm, x_\pm$ and $y'_\pm, x_\pm$ respectively. (Some of $y_\pm, y'_\pm$ might be in $\mathrm{Flag}(\tau_{mod})$.) However, we will assume that the distances $d(x_\pm, y_\pm), d(x'_\pm, y_\pm)$ are sufficiently large, $\geqslant C$.

Lemma 5.10 (Wiggle lemma, II). *Given $\Theta' > \Theta$ there exist $C \geqslant 0$, $\epsilon > 0$ and a Morse datum M' containing Θ' such that if*

$$\mu := \max(\angle^{\zeta}_{x_\pm}(y'_\pm, y_\pm)) < \epsilon,$$

and

$$\nu := \min(d(x_\pm, y_\pm), d(x_\pm, y'_\pm)) \geqslant C$$

then p' is M'-Morse.

Proof. Pick an auxiliary compact Weyl-convex subset Θ_2, $\Theta < \Theta_2 < \Theta'$.

We define biinfinite geodesic extensions $\hat{p}, \hat{p}'$ as in Lemma 5.7, by extending (if necessary) the paths $p_\pm, p'_\pm$ via Θ-Finsler geodesics $y_\pm\tau_\pm$ and $y'_\pm\tau'_\pm$. According to Lemma 5.7, there exists $C > 0$and a Morse datum M_2 (containing Θ_2), both depending on M and Θ_2, such that the path $\hat{p}$ is M_2-Morse. The same lemma applied to the paths $\hat{p}'_\pm$ implies that they are also M_2-Morse.

By the construction,

$$\mu := \angle^\zeta_{x_\pm}(y'_\pm, y_\pm) = \angle^\zeta_{x_\pm}(\tau'_\pm, \tau_\pm).$$

Now, claim follows from Lemma 5.9. □

Lastly, we prove a general Wiggle Lemma where we allow to perturb the entire path p. We consider concatenations

$$p = p_- \star p_0 \star p_+, \quad p' = p'_- \star p'_0 \star p'_+$$

of M-Morse quasigeodesics, where we assume that p_0, p'_0 are within distance D_0 from each other. The paths $p_\pm$ connect $y_\pm, x_\pm$ and $p'_\pm$ connect $y'_\pm, x'_\pm$.

Lemma 5.11 (Wiggle lemma, III). *Given $\Theta' > \Theta$ there exist $C \geqslant 0$, $\epsilon > 0$ and a Morse datum M' containing Θ' such that if*

$$\mu := \max(\angle^\zeta_{x_\pm}(y'_\pm, y_\pm)) < \epsilon,$$

and

$$\nu := \min(d(x_\pm, y_\pm), d(x'_\pm, y'_\pm)) \geqslant C$$

then p' is M'-Morse.

Proof. As before, we fix an auxiliary compact Weyl-convex subset Θ_3, $\Theta < \Theta_3 < \Theta'$. Then $p'_\pm$ are within distance $D_3 = D_3(M, \Theta_3)$ from Θ_3-regular Finsler geodesics $c_\pm := y'_\pm x_\pm$. We apply Lemma 5.10 to the pair of paths

$$p, p'' := c_- \star p_0 \star c_+.$$

It follows that p'' is M_3-Morse for some Morse datum M_3 containing Θ' provided that $\mu \leqslant \epsilon = \epsilon(M, \Theta_3, \Theta')$ and $\nu \geqslant C = C(M, \Theta_3, \Theta')$. Since the paths p'' and p' are wihin distance $D' := \max(D_0, D_3)$ from each other, the path p' is $M' := M_3 + D'$-Morse. □

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Addresses:

M.K.: Department of Mathematics,
University of California, Davis
CA 95616, USA
email: kapovich@math.ucdavis.edu

B.L.: Mathematisches Institut
Universität München
manicures Theresienstr. 39
D-80333, München, Germany
email: b.l@lmu.de

J.P.: Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
08193 Bellaterra, Spain
email: porti@mat.uab.cat