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On the absence of Ahlfors' finiteness theorem for Kleinian groups in dimension three

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Abstract

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We prove the existence of a discontinuous, conformal, finitely generated, function group F which acts freely on the connected component $\Omega \subset \bar{\mathbb{R}}^3$ of domain of discontinuity, such that the group $\pi_1(\Omega/F)$ is not finitely generated.

Keywords: Ahlfors' finiteness theorem, 3-manifold, discontinuous group, Maskit Combination.

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1. Introduction

In the theory of discontinuous Mobius groups acting on the complex plane $\mathbb{C}$ the following strong Ahlfors' finiteness theorem is fundamental for various inquiries [1, 9]:

Let G be a discrete nonelementary finitely generated subgroup of PSL(2, $\mathbb{C}$) acting freely on the domain of discontinuity $\Omega(G)$; then the factor space $\Omega(G)/G$ consists of a finite number of Riemannian surfaces $S_1, \ldots, S_n$ each having a finite hyperbolic area. In particular, the group $\pi_1(S_i)$ is finitely generated ($i = 1, \ldots, n$).

Analytic methods for attacking the finiteness problem for higher-dimensional Kleinian groups were developed in [2, 14]; however these methods fail to shed light on the topology of the factor space of Kleinian groups. In the present paper it will be proved that even a weakened version of Ahlfors' finiteness theorem does not hold in dimension 3:

Theorem. *There exists a finitely generated, torsion free function group* $F \subset \mathrm{Mob}(\bar{\mathbb{R}}^3)$, *with invariant component* $\Omega \subset \Omega(F)$, *such that the group* $\pi_1(\Omega/F)$ *is not finitely generated.*

2. Preliminaries

Let $\mathrm{Mob}(\bar{\mathbb{R}}^n) \simeq \mathrm{Isom}(\mathbb{H}^{n+1})$ be the group of conformal automorphisms of the n-dimensional sphere $S^n = \bar{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, where $\mathbb{H}^{n+1} = \{(x_1, \ldots, x_{n+1}) : x_{n+1} > 0\}$ is the hyperbolic space.

A subgroup $G \subset \mathrm{Mob}(\bar{\mathbb{R}}^n)$ is called Kleinian if the action of G is discontinuous at some point $x \in \bar{\mathbb{R}}^n$, i.e., there exists a neighborhood $U(x)$ such that $g(U(x)) \cap U(x) \neq \emptyset$ only for a finite number of elements $g \in G$. The set of all points at which the action of G is discontinuous is called the domain of discontinuity $\Omega(G)$ and its complement $\Lambda(G) = \bar{\mathbb{R}}^n \setminus \Omega(G)$ is called the limit set of G.

A Kleinian group G is called a function group if there exists a connected component $\Omega \subset \Omega(G)$ which is invariant under G. If G acts freely on Ω, then the factor space $M(G) = \Omega/G$ is an n-dimensional manifold. Let $I(g)$ be the isometric sphere for an element $g \in \mathrm{Mob}(\bar{\mathbb{R}}^n)$. We shall denote by $D(G)$ the isometric fundamental domain [10] for G.

In what follows all manifolds are assumed to have dimension 3 and be piecewise-linear. See [6, 9, 10] for standard material on 3-manifold topology and Kleinian group theory. If $S \subset \mathbb{R}^3$ is a 2-sphere, then we shall denote by ext(S) and int(S) the components of $\bar{\mathbb{R}}^3 \setminus S$ such that $\infty \in \mathrm{ext}(S)$ and $\infty \notin \mathrm{int}(S)$. The symbol cl( ) means the closure of a set.

3. Outline of the proof

Let $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ be mutually tangent euclidean spheres in $\mathbb{R}^3$ (see Fig. 1). Each sphere Σ_i is obtained from a neighboring one by reflection τ_j in the extended

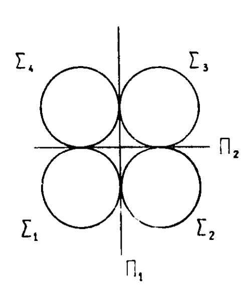

Fig. 1.

euclidean plane Π_j ($i=1,\ldots,4$; $j=1,2$). We shall construct discontinuous groups $\Gamma_i \subset \mathrm{Mob}(\bar{\mathbb{R}}^3)$ which are isomorphic to surface bundle groups and preserve the interiors of the spheres Σ_i, respectively. By using the Maskit combination method we prove that both groups $G_1=\langle\Gamma_1,\Gamma_2\rangle$ and $G_2=\langle\Gamma_3,\Gamma_4\rangle$ are discontinuous and isomorphic to a surface bundle group (see Lemma 3). Let F_i be the corresponding normal surface subgroups of G_i ($i=1,2$). The theorem's proof is finished in Lemma 5, where we show the group $F=\langle F_1, F_2\rangle$ to be the group we need. In particular, it is a normal subgroup of the geometrically finite, function group $G=\langle G_1, G_2\rangle$. The lemma's proof is based on the following considerations. By using the involution τ_2 we construct the manifold $M(F)=\Omega/F$ as the double of some manifold $M(F)^-$. There exists an infinite regular covering $M(F)^- \to M(G)^-$ induced by the covering $M(F)\to M(G)$. The manifold $M(G)^-$ is not a surface bundle since $\partial M(G)^-$ contains a genus 2 surface. It follows that the group $\pi_1(M(F)^-)$ is not finitely generated [6]. We obtain immediately that $\pi_1(M(F))$ is also not finitely generated.

4. Construction of the group F

Let M be an open manifold which is the complement of the Borromean rings. It is well known that M admits a complete hyperbolic structure of finite volume, i.e., $M=\mathbb{H}^3/\Gamma$, $\Gamma\subset \mathrm{Isom}(\mathbb{H}^3)$ [20].

Definition. A group K with a subgroup S is called *S-residually finite* (*S-RF*) if for any element $k\in K\backslash S$ there is a subgroup of finite index $K_1\subset K$ which contains S but does not contain $\{k\}$.

Lemma 1. *The group Γ is S-residually finite for any geometrically finite subgroup S of Γ.*

Proof. Consider a regular ideal octahedron $P\subset\mathbb{H}^3$, all of whose dihedral angles are $\frac{1}{2}\pi$ [20]. Let Q be the reflection group determined by P, and Q_1 be a finite extension of Q by four elements which are automorphisms of order 3. Then Γ is a subgroup of finite index in Q_1 [20]. So the assertion of this lemma follows from [18] and commensurability of Γ and Q. □

Remark. This lemma will be used in the proof of Lemma 2.

Let us consider a model of hyperbolic space which is the exterior B of the unit 3-sphere $\Sigma=\Sigma_1$, centered at zero. Furthermore let H_i be nonconjugate maximal parabolic subgroups of Γ and $\Lambda(H_i)=\{p_i\}$ ($i=1,2$). We assume that the points p_1 and p_2 have coordinates $(0,1,0)$ and $(0,0,1)$ respectively. Let Π_i be the extended euclidean planes tangent to Σ at the points p_i (see Fig. 1) and Π_i^- be a component of $\mathbb{R}^3\backslash\Pi_i$ such that $\Pi_i^- \cap \Sigma = \emptyset$ $(i=1,2)$. Denote by τ_j reflection in the extended euclidean plane Π_j $(j=1,2)$. If A_1 and A_2 are subgroups of $\mathrm{Mob}(S^3)$, then we shall denote by $\langle A_1, A_2\rangle$ the group generated by A_1 and A_2.

In the following lemma we shall prove that all conditions of the Maskit Combination Theorem [5] are satisfied for some subgroup of a finite index $\tilde{\Gamma}\subset\Gamma$ and for the planes Π_i. Consider certain neighborhoods V_i of $\Pi_i\backslash\{p_i\}$. Namely, let W_i be a sphere tangent to Σ at the point p_i such that:

(1) $W_i \subset \mathrm{cl}(B)$;

(2) the ball $\mathrm{cl}(\mathrm{int}(W_i))$ contains Σ.

Put $V_i = \mathrm{ext}(W_i)$ $(i=1,2)$.

Lemma 2. *There exists a subgroup of finite index $\tilde{\Gamma}\subset\Gamma$, for which the following conditions hold.*

(a) *The group $\tilde{\Gamma}$ has a normal finitely generated subgroup $\tilde{F}$ such that $\tilde{\Gamma}=\langle\tilde{F}, t\rangle$, for some $t\in H_2\cap\tilde{\Gamma}$.*

(b) *The group $\tilde{\Gamma}$ has a fundamental set $\mathcal{P}$ such that $\mathcal{P}\cap V_i$ is a fundamental domain for the action of the group $\tilde{H}_i = H_i\cap\tilde{\Gamma}$ on V_i $(i=1,2)$.*

Proof. By compactness arguments, there exists at most a finite number of elements $h_k\in H_i$ such that $I(h_k)\cap(\Pi_j\cup(\bar{\mathbb{R}}^3\backslash D(H_j)))\neq\emptyset$ $(j\neq i;\ i,j\in\{1,2\}, 0\leq k<\infty)$. According to the residually finiteness of the group Γ [11] we can choose a subgroup of finite index $\hat{\Gamma}\subset\Gamma$ for which $h_k\notin\hat{\Gamma}$. Let $\hat{H}_i=\hat{\Gamma}\cap H_i$ $(i=1,2)$.

We now prove (a). Let Φ be a normal subgroup of Γ which corresponds to a fiber of M; then $\hat{\Phi}=\Phi\cap\hat{\Gamma}$ is a normal subgroup of $\hat{\Gamma}$. There exists $l\in\hat{\Gamma}$ such that $\hat{\Gamma}=\langle\hat{\Phi}, l\rangle$. Let $t\in\hat{H}_2\backslash\hat{\Phi}$ then $t=\alpha\cdot l^n$, for some $\alpha\in\hat{\Phi}$. Set $\Gamma^0=\langle\hat{\Phi}, l^n\rangle=\langle\hat{\Phi}, t\rangle$. Clearly we have $|\Gamma:\Gamma^0|<\infty$.

Now we prove (b). Let $\tilde{H}_i=\hat{H}_i\cap\Gamma^0$. We have proved that $\mathrm{cl}(\bar{\mathbb{R}}^3\backslash D(\hat{H}_i))\subset D(\hat{H}_j)$, for $i\neq j$; then by the Klein Combination Theorem the set $D(\tilde{H}_1)\cap D(\tilde{H}_2)$ is a fundamental domain for the Schottky-type group $\tilde{H}=\langle\tilde{H}_1,\tilde{H}_2\rangle=\tilde{H}_1*\tilde{H}_2$ [10]. Hence the set $R=D(\tilde{H}_1)\cap D(\tilde{H}_2)\cap\mathrm{cl}(V_1\cup V_2)$ has no equivalent points for action of $\tilde{H}$. The closure of the set $T=R\cap(W_1\cup W_2)$ is compact in B; hence there is at most a finite number of elements $g_k\in\Gamma^0$ such that $g_k(T)\cap T\neq\emptyset$ $(k=1,\ldots,m)$. The group $\tilde{H}$ is geometrically finite [10]; hence Γ^0 is $\tilde{H}$-residually finite according to Lemma 1. Thus there exists a subgroup $\tilde{\Gamma}\subset\Gamma^0$ such that $|\Gamma^0:\tilde{\Gamma}|<\infty$, $\tilde{H}\subset\tilde{\Gamma}$ and $g_k\notin\tilde{\Gamma}$, $k=1,\ldots,m$. It is clear that for any $g\in\tilde{\Gamma}$ we have $g(R)\cap R=\emptyset$. Indeed, suppose there exists an element $g\in\tilde{\Gamma}$ for which the last assertion is not valid. Then $g\notin\tilde{H}$ because R does not contain equivalent points under $\tilde{H}$. Further, there are h_1, $h_2\in\tilde{H}_1\cup\tilde{H}_2$ such that $h_2\cdot g\cdot h_1(T)\cap T\neq\emptyset$. This is impossible since $\gamma=h_2\cdot g\cdot h_1\neq 1$, $\gamma\in\tilde{\Gamma}$. Clearly the group $\tilde{\Gamma}$ satisfies condition (a) as well. So R is a fundamental set for the action of $\tilde{\Gamma}$ in the orbit $\tilde{\Gamma}(V_1\cup V_2)$.

Choose an arbitrary fundamental set A for action of $\tilde{\Gamma}$ in $B\backslash\tilde{\Gamma}(\mathrm{cl}(V_1\cup V_2))$. Hence $A\cup R=\mathcal{P}$ is a fundamental set for action of the group $\tilde{\Gamma}$ in B. For this fundamental set the condition (b) holds.

Finally put $\tilde{F} = \tilde{\Gamma} \cap \Phi$. □

Let us introduce notations: $\Gamma_1 = \tilde{\Gamma}$, $\Gamma_2 = \tau_1 \Gamma_1 \tau_1$, $G_1 = \langle \Gamma_1, \Gamma_2 \rangle$, $G_2 = \tau_2 G_1 \tau_2$, $G = \langle G_1, G_2 \rangle$.

Lemma 3. *The group G_1 is discontinuous and contains a finitely generated normal subgroup F_1 such that $G_1/F_1 \simeq \mathbb{Z}$ and $G_1 = \langle F_1, t \rangle$, where t is the element defined by Lemma 2.*

Proof. Let $\mathscr{P}_1 = \mathscr{P}$, where $\mathscr{P}$ is the fundamental set of the group $\tilde{\Gamma} = \Gamma_1$ constructed in Lemma 2. The group $\tilde{H}_1 = \Gamma_1 \cap \Gamma_2$ stabilizes the plane Π_1. According to assertion (b) of Lemma 2 and maximality of the parabolic subgroups $\tilde{H}_i$ of Γ_1, the domain $\mathrm{cl}(\Pi_1^-)$ is precisely invariant under $\tilde{H}_1$ in the group Γ_1. By analogy the domain $\tau_1\ \mathrm{cl}(\Pi_1^-)$ is precisely invariant under the subgroup $\tilde{H}_1$ of Γ_2. Thus all conditions of Maskit Combination 1 Theorem [13] are satisfied (the multidimensional version of the Combination Theorem is in [5]). Consequently, the group G_1 is discontinuous, isomorphic to $\Gamma_1 *_{\tilde{H}_1} \Gamma_2$, and has as its fundamental set $R_1 = \mathscr{P}_1 \cap \tau_1(\mathscr{P}_1)$. Moreover the group G_1 acts on the invariant component Ω_1 ($\infty \in \Omega_1$).

Claim (see also [15]). *The manifold $M(G_1) = \Omega_1/G_1$ is homeomorphic to the interior of a surface bundle over S^1 and $\pi_1(\Omega_1) = 1$.*

Proof. From geometric decomposition of the group $G_1 = \Gamma_1 *_{\tilde{H}_1} \Gamma_2$ it follows that $M(G_1)$ is obtained by glueing of M_1 and M_2, where $M_1 = M(\Gamma_1) \backslash (\Pi_1^-/\tilde{H}_1)$, $M_2 = M(\Gamma_2) \backslash (\tau_1 \Pi_1^-/\tilde{H}_1)$. Furthermore $\Pi_1^-/\tilde{H}_1 = \tau_1 \Pi_1^-/\tilde{H}_1 = S^1 \times S^1 \times (0, 1)$. Therefore each manifold M_i is homeomorphic to a surface bundle whose fiber correspond to the subgroup $\tilde{F} \subset \tilde{\Gamma}$. The glueing homeomorphism $\bar{\varphi} : \partial M_1 \to \partial M_2$ preserves the bundle structure since it is covered by the identity homeomorphism $\varphi : \Pi_1 \to \Pi_1$. By van Kampen's theorem, we have the isomorphism $\pi_1(M(G_1)) \simeq \Gamma_1 *_{\tilde{H}_1} \Gamma_2 \simeq G_1$. The group G_1 is a Hopfian group [11]; hence $\pi_1(\Omega_1) = 1$. The claim is proved.

Thus the subgroup F_1 of G_1 which corresponds to a fiber of $M(G_1)$ is a normal subgroup and $G_1/F_1 \simeq \mathbb{Z}$. Due to assertion (a) of Lemma 2 we also have $G_1 = \langle F_1, t \rangle$ where $t \in \tilde{H}_2$. Each surface bundle $M(\Gamma_i)$ admits a natural compactification by adjoining a torus for each cusp, and the same is true for $M(G_1)$. Hence the group F_1 is finitely generated. □

We set $F = \langle F_1, F_2 \rangle$, where $F_2 = \tau_2 F_1 \tau_2$. Let $\tilde{H}_3 = \tau_1 \tilde{H}_2 \tau_1$ and $J = \langle \tilde{H}_2, \tilde{H}_3 \rangle$.

Lemma 4. (a) *The group G is the Maskit Combination of the groups G_1 and G_2 along the subgroup J.*

(b) *The group G is discontinuous and has an invariant component Ω (which we take to be the component containing ∞).*

(c) *The finitely generated group F is a normal subgroup in G.*

(d) *The manifold $M(G) = \Omega/G$ is the interior of a compact manifold.*

Proof. (a) By virtue of Lemma 3, the group G_1 acts discontinuously on Ω_1 and $R_1 = \mathcal{P} \cap \tau_1(\mathcal{P})$ is a fundamental set for this action. Due to Lemma 2 the domain $R_1 \cap \mathrm{cl}(\Pi_2^-)$ is a fundamental set for the action of the group J on the ball $\mathrm{cl}(\Pi_2^-)$. Moreover in the neighborhood $V = V_2 \cap \tau_1(V_2)$ of $\Pi_2 \backslash \Lambda(J)$ we have $R_1 \cap V = D(\tilde{H}_2) \cap D(\tilde{H}_3) \cap V$ (see Lemma 2) and the open surface $\Pi_2 \backslash \Lambda(J)$ is precisely invariant under J in the group G_1. Hence there exists a neighborhood $\mathcal{N}$ of $\Pi_2 \backslash \Lambda(J)$ such that $\mathcal{N}$ is also precisely invariant under J in the group G_1 and $\mathcal{N} \subset \Omega(G_1)$.

To prove assertion (a) it remains to verify the following claim.

Claim. *The sphere Π_2 is precisely invariant under the subgroup J of G_1.*

Proof. Let us suppose that there exists an element $g \in G_1 \backslash J$ for which $g(\Pi_2) \cap \Pi_2 = \{x\} \subset \Lambda(J)$. The Schottky-type group J is geometrically finite [10, 12]; and we have two cases [4]:

(1) x is a point of approximation, or

(2) x is fixed point of a parabolic transformation $\gamma \in J$.

In the case (1), there exists a sequence $h_n \in J$ such that $\lim_{n\to\infty} h_n(x) = x_0$ and $y_0 = \lim_{n\to\infty} h_n(z) \neq x_0$ for any $z \in \mathrm{cl}(\Pi_2^-) \backslash \{x\}$, where $y_0 \in \Pi_2$. It is easy to see that the sequence of spheres $h_n g(\Pi_2)$ converges to Π_2. Therefore the intersection $h_n g(\mathcal{N}) \cap \mathcal{N}$ is not empty for large n. This is impossible, since $\mathcal{N}$ is precisely invariant under J in the group G_1 and all elements h_n are different.

In the case (2), there exist elements $h, h' \in J$ such that $h \circ g \circ h'(\{p_2, p_3\}) = \{p_2, p_3\}$, where $p_3 = \tau_1(p_2)$. Due to the fact that $\tilde{H}_2$, $\tilde{H}_3$ are maximal and nonconjugate parabolic subgroups of G_1, the element g belongs to J, which is impossible. The claim is proved.

We immediately obtain assertion (b) of Lemma 4 from (a) via the First Maskit Combination Theorem.

To prove assertion (c) we have to verify that for any $g_1 \in G_1$ and $g_2 \in G_2$ the relations $g_1^{-1} F_2 g_1 \subset F = \langle F_1, F_2 \rangle$ and $g_2 F_1 g_2^{-1} \subset F$ hold. The element g_1 has the form $f_1 t^n$, where $f_1 \in F_1$, $t \in \tilde{H}_2 \subset G_1 \cap G_2$, $G_2 = \langle F_2, t \rangle$ (see Lemma 3). So, $g_1 F_2 g_1^{-1} = f_1 t^n F_2 t^{-n} f_1^{-1} = f_1 F_2 f_1^{-1} \subset \langle F_1, F_2 \rangle$.

Thus assertion (c) is proved.

(d) As we have seen in Lemma 3, both manifolds $M(G_1)$ and $M(G_2)$ admit natural compactifications by adjoining a torus for each cusp. Hence both manifolds $M(G_1)^- = M(G_1) \backslash \Pi_2^- / J$ and $M(G_2)^- = M(G_2) \backslash (\tau_2(\Pi_2^-)/J)$, as well as the manifold $M(G)$, which is obtained by glueing them along the compact surface $\bar{S} = (\Pi_2 \backslash \Lambda(J))/J$, admit compactifications as compact manifolds with boundary. □

Notice that $\Lambda(F) = \Lambda(G)$ since F is normal subgroup of G. Moreover, according to assertions (b) and (c) of Lemma 4, we have the groups G and F possess a common invariant component Ω. Let $M(F)$ be the manifold Ω/F.

Lemma 5. *The group* $\pi_1(M(F))$ *is not finitely generated.*

Proof. *Step* 1. First let us verify that the orbits $G_1(\Pi_2^-)$ and $F_1(\Pi_2^-)$ are equal. Indeed, $G_1 = \langle F_1, t\rangle$, where $t \in \tilde{H}_2$. Thus, since $t(\Pi_2^-) = \Pi_2^-$, the equality $G_1(\Pi_2^-) = F_1(\Pi_2^-)$ holds. Further we claim that G is generated by F and t. Indeed, for each $g \in G$ we have the decomposition $g = g_1 g_2 \cdots g_n$ (where $g_j \in G_1 \cup G_2$); hence from the equalities $g_j = f_j t^{m_j}$ ($f_j \in F_1 \cup F_2$) we obtain $g = ft^m$, where $f \in F$ and $m \in \mathbb{Z}$.

Note: We do not yet claim that $G/F \simeq \mathbb{Z}$. This isomorphism will be established later.

Step 2. By construction we have $\tau_2 G \tau_2 = G$. Therefore by means of the covering $p: \Omega \to \Omega/G = M(G)$, the involution τ_2 projects to the involution $\bar{\tau}_2: M(G) \to M(G)$. Clearly the surface $\bar{S} = p(\Pi_2 \backslash \Lambda(J))$ is the fixed-point set for the last involution. In the same manner the involution τ_2 projects to the involution $\hat{\tau}_2: \Omega/F \to \Omega/F = M(F)$. So we have the commutative diagram:

$$\begin{array}{ccccc} \Omega & \xrightarrow{q} & M(F) & \xrightarrow{r} & M(G) \\ \downarrow{\scriptstyle \tau_2} & & \downarrow{\scriptstyle \hat{\tau}_2} & & \downarrow{\scriptstyle \bar{\tau}} \\ \Omega & \xrightarrow{q} & M(F) & \xrightarrow{r} & M(G) \end{array}$$

where $p = r \circ q$ and r is a regular covering with the deck-transformation group G/F. The surface $\hat{S} = r^{-1}(\bar{S}) = q(\Pi_2 \backslash \Lambda(J))$ is a connected surface (due to Step 1). Clearly $\hat{S}$ is the fixed-point set for the $\hat{\tau}_2$.

Step 3. Since the group G results from the Maskit Combination of the groups G_1 and G_2, the domain $(\Omega_1 \backslash (G_1(\Pi_2^-)))/G_1$ is the closure of some component of $M(G)\backslash \bar{S}$. Let us denote this closure by $M(G)^-$. Let $M(F)^-$ be the manifold $r^{-1}(M(G)^-)$. On the other hand, the manifolds $M(F)^-$ and $M(G)^-$ are equal to $M(F_1)\backslash(\Pi_2^-/J \cap F)$ and $M(G_1)\backslash(\Pi_2^-/J)$, respectively. Thus the covering $r: M(F)^- \to M(G)^-$ is just the restriction of the infinite cyclic covering $M(F) \to M(G)$.

Step 4. As we have seen in Lemma 4, the manifold $M(G)^-$ may be compactified as $N(G)^-$. The boundary component $\bar{S}$ of $M(G)^-$ is a compact surface of genus 2 (it is the quotient of plane domain $\Pi_2 \cap \Omega(J)$ by the Schottky-type group J). Hence the manifold $N(G)^-$ is not a surface bundle over S^1. Moreover both manifolds $N(G)^-$ and $N(F)^-$ do not contain fake cells since they are covered by subdomains of S^3.

Step 5. Here we shall prove that the group $\pi_1(M(F)^-)$ is not finitely generated. Due to Step 3 we have the exact sequence

$$1 \to \pi_1(M(F)^-) \to \pi_1(M(G)^-) \simeq \pi_1(N(G)^-) \to \mathbb{Z} \to 1.$$

Let us suppose that the group $\pi_1(M(F)^-)$ is finitely generated. The manifold $M(G)^-$ does not contain projective planes, due to the $\mathbb{R}P^2$-irreducibility of the manifold

$M(G_1)$. Furthermore $\pi_1(M(F)^-)$ is a non-Abelian group; hence by Stallings' theorem [6, Theorem 11.6] the manifold $N(G)^-$ is homeomorphic to a surface bundle over S^1. However this contradicts Step 4.

So the group $\pi_1(M(F)^-)$ is not finitely generated.

Step 6. It remains to prove that the group $\pi_1(M(F))$ also is not finitely generated. Let $u:\tilde{M}\to M(F)$ be the universal covering with automorphism group $\pi \simeq \pi_1(M(F))$. Evidently the manifold $M(F)^-$ is homeomorphic to $M(F)/\hat{\tau}_2$. Let us consider a lift $\tilde{\tau}_2:\tilde{M}\to\tilde{M}$ of the involution $\hat{\tau}_2$. Thus $\pi=\tilde{\tau}_2\pi\tilde{\tau}_2$ and the group $\mathfrak{G}=\langle\pi_1(M(F)),\tilde{\tau}_2\rangle$ acts discontinuously on $\tilde{M}$. Let TORS be the normal subgroup of $\mathfrak{G}$ generated by its elements of finite order. Then by Armstrong's theorem [3], $\pi_1(M(F)^-)$ is isomorphic to $\mathfrak{G}$/TORS; hence the group $\mathfrak{G}$ is not finitely generated. Evidently the group $\pi_1(M(F))$ is not finitely generated also (as a subgroup of index 2). By construction, the conformal group $F=\langle F_1, F_2\rangle$ is finitely generated, hence Lemma 5 and the theorem are proved. □

5. Concluding remarks

As noticed by the second author [16, 8], the finitely generated group F constructed in the theorem is not finitely presented.

By related ideas the first author [7, 8] showed that multidimensional versions of Ahlfors' and Sullivan's [19] finiteness theorems do not hold. Namely, there exists a finitely generated free Kleinian group $K_3\subset \text{Mob}(S^3)$ such that

(a) the number of conjugacy classes of maximal parabolic subgroups of K_3 is infinite;

(b) if $K_n\subset\text{Mob}(S^n)$ is the conformal extension of K_3 to S^n ($n\geqslant 3$), then

$$\text{rank}(H_{n-1}(\Omega(K_n)/K_n, \mathbb{Q}))=\infty.$$

The manifold $M(K_n)=\Omega(K_n)/K_n$ has infinite homotopy type.

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