GEOMETRY OF QUASI-PLANES

MICHAEL KAPOVICH AND BRUCE KLEINER

Abstract. In this paper we discuss metric cell complexes satisfying a coarse form of 2-dimensional Poincaré duality. We prove that such spaces are either Gromov-hyperbolic or have polynomial growth. As an application we prove that 2-dimensional Poincaré duality groups over commutative rings are commensurable with surface groups.

1. Introduction

This paper is the second in a sequence of three papers which deal with spaces satisfying coarse Poincaré duality [14] in conjunction with a coarse generalization of the Seifert fibered space conjecture [13]. In the present paper we study geometry of *quasi-planes*, i.e. simply-connected metric cell complexes satisfying *coarse Poincaré duality* (over a commutative ring $\mathcal{R}$) in dimension 2. We refer the reader to section 3 for the precise definition, at the moment we only note that quasi-planes are metric spaces satisfying a coarse version of the Jordan separation theorem (for curves in $\mathbb{R}^2$). We introduce a *surrounding function* for quasi-planes and prove

Theorem 1.1. *Suppose that Z is a quasi-homogeneous quasi-plane. Then we have the following dichotomy:*

1. Either the surrounding function is super-linear, in which case Z is Gromov-hyperbolic, with topological circle as the ideal boundary. In this case each cobounded quasi-action $G \curvearrowright Z$ is quasi-isometrically conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.

2. Or the surrounding function is at most linear, in which case Z has polynomial growth and, in case Z is quasi-isometric to a finitely generated group Q, the group Q is virtually abelian of rank 2.

We note that in the case when Z is the zero-skeleton of a 2-dimensional triangulated planar surface S, such that S is quasi-isometric to a finitely generated group G, the above theorem was established by G. Mess in [19]. His paper was an inspiration for our work.

As an application we get the following characterization of 2-dimensional Poincaré duality groups over commutative rings, originally established in [16] using somewhat different methods:

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Theorem 1.2. *Suppose that $\mathcal{R}$ is a commutative ring with a unit, G is a 2-dimensional Poincare duality group over $\mathcal{R}$. Then G is commensurable to a surface group.*

This theorem extends earlier results of Eckmann, Müller and Linnel [8, 7], in the case $\mathcal{R} = \mathbb{Z}$, and results of Bowditch [3], in the case when $\mathcal{R}$ is a field. Theorem 1.2 was conjectured by Dicks and Dunwoody in [5]. The key to the proof of this theorem is a construction of an action of G on a quasi-plane Z (over $\mathcal{R}$), which then allows us to apply Theorem 1.1 to conclude that either G is virtually nilpotent or G acts discretely cocompactly isometrically on the hyperbolic plane.

In order to prove Theorems 1.1 and 1.2 we establish a local-to-global characterization of Gromov-hyperbolic spaces. As an application of this characterization we also get a theorem of independent interest:

Theorem 1.3. *Suppose that G is a finitely presented group such that* some[1] *asymptotic cone of G is a metric tree. Then G is Gromov-hyperbolic.*

This contrasts with the fact that there are examples [22], [6], of finitely generated groups G for which *some* asymptotic cone of G is a metric tree but G is *not Gromov-hyperbolic.*

In the forthcoming paper [13] we will apply our results to study *coarse fibrations* of manifolds by lines. In particular, we prove a coarse analogue of the Seifert fibered space conjecture and give an alternative proof of G. Mess' part of the proof of the original Seifert fibered space conjecture.

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2. Preliminaries

2.1. **Definitions and notation.** We let $\mathbb{Z}_+ := \{m \in \mathbb{Z} \mid m \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. We let $d_H(\cdot,\cdot)$ denote the Hausdorff distance between subsets of a metric space; the usual (infimal) distance will be denoted $d(\cdot,\cdot)$. All maps between cell complexes will be continuous unless otherwise specified. Given a map f we let $\mathrm{Im}(f)$ denote the image of f.

Throughout this paper we fix a commutative ring $\mathcal{R}$ with a unit and we will be using singular (co)homology with coefficients in $\mathcal{R}$ unless we indicate otherwise. For each negative integer k we set $H_k(\cdot) = 0$, $H_c^k(\cdot) = 0$, etc.

Let r_i, R_i be two sequences of positive real numbers. We will use the notation

$$R_i \gtrsim r_i$$

[1]It is easy to see that if *all* asymptotic cones of a finitely-generated group G are metric trees, then G is Gromov-hyperbolic.

if there exist a pair of constants A, B (independent of i) such that for all but finitely many $i \in \mathbb{N}$ we have:

$$R_i \geq Ar_i + B.$$

We will use the notation $R_i \simeq r_i$ if $R_i \gtrsim r_i$ and $r_i \gtrsim R_i$.

A subset $S \subset Z$ of a metric space is called δ-*dense* if each point $z \in Z$ is within distance $\leq \delta$ from S. A subset in Z which is δ-dense from some $\delta < \infty$, is called a *net* in Z.

A metric space Z has *bounded geometry* if there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each metric ball $B(x, r) \subset Z$, one has:

$$V(x, r) := |B(x, r)| \leq \phi(r),$$

where $|S|$ denotes the cardinality of a set S.

For a subset $D \subset Z$ define the *c-frontier* $\partial_c D$ as

$$\partial_c D := \{x \in Z \setminus D : d(x, D) \leq c\}.$$

For $c = 1$ we set $\partial := \partial_1$.

2.2. **Maps and actions.** A map $f : X \to X'$ between metric spaces is *uniformly proper* if there are constants L, A, and a continuous strictly increasing *distortion function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \eta(t) = \infty$ such that

$$\eta(d(x_1, x_2)) \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A \tag{2.1}$$

for all $x_1,\ x_2 \in X$.

A map $f : X \to X'$ between metric spaces X and X' is an (L, A)-*quasi-isometry* if for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) + A$$

and $\mathrm{Im}(f)$ is A-dense in X'. Here $L \geq 1$ and $A \in \mathbb{R}$. We let $\widehat{\mathrm{QI}}(X, X')$ denote the collection of all quasi-isometries from X to X'. Two quasi-isometries $f_1, f_2 : X \to X'$ are *equivalent* if $d(f_1, f_2) < \infty$; we let $\mathrm{QI}(X, X')$ denote the set of equivalent classes of quasi-isometries, and use $\mathrm{QI}(X)$ (resp. $\widehat{\mathrm{QI}}(X)$) in place of $\mathrm{QI}(X, X)$ (resp. $\widehat{\mathrm{QI}}(X, X)$). Composition of quasi-isometries induces a group structure on $\mathrm{QI}(X)$.

Remark 2.2. Suppose that f_i are $(L_i, L_i - 1)$-quasi-isometries, $i = 1, 2$. Then their composition $f_2 \circ f_1$ is an $(L_1L_2, L_1L_2 - 1)$-quasi-isometry. Therefore, if $L(f)$ denotes the logarithm of infimal $L \geq 1$ such that the mapping f is an $(L, L-1)$-quasi-isometry, then

$$L(f_2 \circ f_1) \leq L(f_1) + L(f_2).$$

A *quasi-action* of a group G on a metric space X, denoted $G \overset{\rho}{\curvearrowright} X$, is a map $\rho : G \to \widehat{\mathrm{QI}}(X)$ such that for suitable constants L, A,

1. $\rho(g)$ is an (L, A)-quasi-isometry for all $g \in G$,
2. $d(\rho(1), \mathrm{id}_X) < A$, and
3. $d(\rho(g_1 g_2), \rho(g_1)\rho(g_2)) < A$ for all g_1, $g_2 \in G$.

We will usually write $g(x)$ rather than $\rho(g)(x)$, suppressing the name of the quasi-action when it is understood.

A quasi-action $G \overset{\rho}{\curvearrowright} X$ is called *cobounded* if for some (for every) point $x \in X$ the *quasi-orbit* $G \cdot x = \{g(x) : g \in G\}$ is a net in X. We say that a metric space X is *quasi-homogeneous* if there are constants L, A such that for all x, $x' \in X$ there is an (L, A) quasi-isometry $f : X \to X'$ with $f(x) = x'$. Note that if there exists a cobounded quasi-action $G \curvearrowright X$ then X is quasi-homogeneous.

Lemma 2.3. *Suppose that Y is a proper geodesic metric space, $H \curvearrowright Y$ is an (L, A)-quasi-action. Then there exists $\epsilon \ll L$ and R_0 such that if for $R \geq R_0$, $B(y, R) \cap H(y)$ is an ϵR-net in $B(y, R)$, then the quasi-action is cobounded.*

Proof. Suppose $R < \infty$, $\epsilon > 0$, and $H(y)$ is an ϵR-net in $B(y, R)$. There is a constant $\lambda > 0$ depending only the geometry of the quasi-action $H \curvearrowright Y$ such that for each $y' \in H(y)$, the quasi-orbit $H(y)$ forms a $\frac{\epsilon}{\lambda} R$ net in $B(y', \lambda R)$. Thus when ϵ is sufficiently small and R is sufficiently large, then for all $z \in Y$, if we choose $y' \in H(y)$ such that $d(z, y') \leq d(z, H(y)) + 1$, then we conclude that $d(z, H(y))$ is uniformly bounded; otherwise the geodesic segment $\overline{y'z}$ would have to pass within distance $\frac{\epsilon}{\lambda} R$ of a point after traveling a distance $\frac{\lambda}{2} R$ from y'. □

2.3. **Gromov hyperbolicity.** Let Z be a geodesic metric space. A geodesic triangle $\Delta \subset Z$ is called *R-thin* if every side of Δ is contained in the R-neighborhoodof the union of two other sides. An *R-fat triangle* is a geodesic triangle which is not R-thin. A geodesic metric space Z is called *δ-hyperbolic in the sense of Rips* (Rips was the first to introduce this definition) if each geodesic triangle in Z is δ-thin.

Let X be a metric space (which is no longer required to be geodesic). For each base-point $p \in X$ define a number $\delta_p \in [0, \infty]$ as follows. For each $x \in X$ set $|x|_p := d(x, p)$ and

$$(x, y)_p := \frac{1}{2}(|x|_p + |y|_p - d(x, y)).$$

Then

$$\delta_p := \inf_{\delta \in [0,\infty]} \{\delta | \forall x, y, z \in X, (x, y)_p \geq \min((x, z)_p, (y, z)_p) - \delta\}.$$

We say that X is *δ-hyperbolic in the sense of Gromov*, if $\infty > \delta \geq \delta_p$ for some $p \in X$. We note that if X a geodesic metric space which is δ-hyperbolic in Gromov's sense then X is 4δ-hyperbolic in the sense of Rips and vice-versa (see [11, 6.3C]).

A metric space Z is *Gromov-hyperbolic* if it is δ-hyperbolic for some $\delta < \infty$.

2.4. **Growth of spaces.** A bounded geometry metric space Z has *polynomial growth* if there is a constant $c \in \mathbb{R}_+$ such that for each ball $B(x,r) \subset X$ one has

$$|B(x,r)| \lesssim r^c, r \in \mathbb{N}.$$

The optimal constant c is called the *degree of the polynomial growth.* A space Z has *superpolynomial growth* if it does not have polynomial growth for any c.

Similarly, a bounded geometry metric space Z has *exponential growth* if there is a constant $a > 0$ such that for each $x \in Z$ one has:

$$|B(x,R)| \gtrsim e^{aR}, r \in \mathbb{N}.$$

A space has *subexponential growth* if

$$|B(x,r)| \lesssim e^{aR}, r \in \mathbb{N},$$

for all $a > 0$.

A metric space Z is N-*doubling* for a certain $N \in \mathbb{R}$ if each ball $B(x,2R) \subset Z$ (where $R \geq 1$) is contained in the union of $\leq N$ balls of radius R. A space is called *doubling* if it is N-doubling for some N, we will refer to N as a *doubling constant* of X.

Lemma 2.4. *Suppose that Z is a bounded geometry doubling metric space. Then Z has polynomial growth.*

Proof. For each $r \geq 1$ choose $n \in \mathbb{Z}$ such that $n - 1 \leq \log_2(r) \leq n$. Hence for each metric ball $B(x,r) \subset X$ we have:

$$B(x,r) \subset \bigcup_{i=1}^{N} B(x_i, \frac{r}{2}) \subset \bigcup_{j}^{N^2} B(x_j, \frac{r}{4}) ... \subset \bigcup_{\ell=1}^{N^{n-1}} B(x_\ell, r2^{-(n-1)}).$$

Let $M := \sup_{x \in Z} |B(x,2)|$ and $c := \frac{1}{\log_N(2)}$. Then

$$|B(x,r)| \leq N^{n-1} M \leq N^{c \log_N(r)} M = r^c M. \quad \square$$

Theorem 2.5 (M. Gromov, [10])**.** *Suppose that G is a finitely-generated group of polynomial growth. Then G is virtually nilpotent.*

An improvement of this theorem was established by L. Van den Dries and A. Wilkie [23]:

Theorem 2.6. *Suppose that G is a finitely-generated group and there is a sequence of metric balls $B(x_i, R_i) \subset G$ and constant $c \geq 0$ such that $|B(x_i, R_i)| \lesssim R_i^c$. Then G is virtually nilpotent and has polynomial growth of degree $\leq c$.*

2.5. **Chain recurrence.** Suppose that Z is a topological space and we are given a continuous $\mathbb{R}$-action $Z \times \mathbb{R} \to Z$. The *$\omega$-limit set* $\omega(z)$ of a point $z \in Z$ is defined as the set of points $\lambda \in Z$ such that there exists a sequence $t_i \in \mathbb{R}$ diverging to $+\infty$ such that

$$\lim_i t_i(z) = \lambda.$$

The set Λ is clearly $\mathbb{R}$-invariant. We will also need the following definition:

Definition 2.7. [Chain recurrence] Suppose we have a continuous $\mathbb{R}$-action on a compact metric space Z. A point $z \in Z$ is called *chain recurrent* if for each $\epsilon > 0$ and $T < \infty$ there exists a finite sequence $z = x_1, ..., x_k = z$, where for each $i = 2, ..., k$, there exists $t > T$ such that

$$d(x_i, t(x_{i-1})) < \epsilon.$$

We recall the following standard lemma from the dynamical system theory:

Lemma 2.8 (See [9], Theorem A). *Let $\mathbb{R} \curvearrowright Z$ be a continuous $\mathbb{R}$-action on a compact metric space Z. Then for each $z \in Z$ and $z' \in \omega(z)$, the point z' is chain-recurrent in the restricted dynamical system $\mathbb{R} \curvearrowright \omega(z)$.*

2.6. **Gromov-Hausdorff convergence and asymptotic cones.** Let $\mathcal{X}$ denote the set of complete pointed metric spaces (X, x). Gromov-Hausdorff topology on $\mathcal{X}$ is defined as follows:

(X, x, H) and (X', x', H') are ϵ-close if there are maps

$$f : (B(x, \frac{1}{\epsilon}), x) \to (X', x')$$

and

$$f' : (B(x', \frac{1}{\epsilon}), x) \to (X', x')$$

which are (e^ϵ, ϵ) quasi-isometric embeddings such that

Axiom 1. $d(f \circ f', Id) \le \epsilon$, $d(f' \circ f, Id) \le \epsilon$.

Define a subset $\mathcal{X}_\Delta \subset \mathcal{X}$ which consists of Δ-doubling pointed metric spaces. Then $\mathcal{X}_\Delta$ is compact (see [10]).

A general sequence in $\mathcal{X}$ does not contain a convergent subsequence. There is however a concept which allows one to construct a limit in this case as well, i.e, an *ultralimit* of a sequence of pointed metric spaces

$$\lim_\eta (X_i, x_i) = (X_\eta, x_\eta),$$

where η is a nonprincipal ultrafilter on $\mathbb{N}$. We refer the reader to [15, 17] for the definition and properties of this construction. If (X_i, x_i) converges to (X, x) in the pointed Gromov-Hausdorff topology then for each nonprincipal ultrafilter η

$$\lim_\eta (X_i, x_i) = (X, x),$$

see for instance [15]. Using ultralimits one defines *asymptotic cones* of a metric space as follows. Suppose that (X, d) is a metric space, $x_i \in X$ is a sequence of base-points, λ_i is a sequence of positive numbers converging to zero. Pick a nonprincipal ultrafilter η on $\mathbb{N}$. Given this data the corresponding asymptotic cone of X is the ultralimit:

$$Cone_\eta(X) := \lim_\eta(X, \lambda_i d, x_i).$$

The concept of Gromov-Hausdorff convergence generalizes in the context of *quasi-actions* on pointed metric spaces:

Fix a doubling constant $\Delta < \infty$, and quasi-action constants L, A. Consider the collection $\mathcal{M}$ of triples (X, x, H), where X is a complete Δ-doubling metric space, $x \in X$, and $H \subset \mathrm{QI}(X)$ is a collection of (L, A)-quasi-isometries with the property that:

For every $h_1, h_2 \in H$, there is an $h_3 \in H$ such that $d(h_3, h_1 \circ h_2) \le A$.

Then (X, x, H) and (X', x', H') are ϵ-close if there are maps f, f' as in **Axiom 1** above so that in addition we have:

Axiom 2. For each $h \in H$ there exists $h' \in H'$ such that for $g' := f \circ h \circ f'$ we have

$$d(g'|B(x', \frac{1}{2\epsilon}), h'|Dom(g') \cap B(x', \frac{1}{2\epsilon})) \le \epsilon.$$

Axiom 3. For each $h' \in H'$ there exists $h \in H$ such that for $g := f' \circ h' \circ f$ we have

$$d(g|B(x, \frac{1}{2\epsilon}), h|Dom(g) \cap B(x, \frac{1}{2\epsilon})) \le \epsilon.$$

Lemma 2.9. *The space $\mathcal{M}$ is compact and metrizable.*

Proof. The first assertion follows from compactness of $\mathcal{X}_\Delta$ and the Arzela-Ascoli theorem. The second assertion follows from Remark 2.2. $\square$

Observe that the group $\mathbb{R}$ acts on $\mathcal{M}$ by scaling: $t \in \mathbb{R}$ scales the metric on X by e^{-t} and does not change the base-point and the collection H of quasi-isometries.

Therefore for each $m \in \mathcal{M}$, the ω-limit set $\omega(m)$ is the collection of all asymptotic cones of m.

2.7. **Metric cell complexes.** We will be working with CW complexes endowed with an extra structure. Let X be a CW complex, and $X^{(m)}$ denote its m-skeleton, $m \in \mathbb{Z}_+$. *We will not assume that X is finite-dimensional,* this degree of generality will be important for the applications of our results presented in [13]. Recall that a subcomplex Y of X is a closed subset which is a union of open cells, such that the boundary of each open cell $\sigma \subset Y$ is contained in Y.

A *control map* for X is a function $p : X \to X^{(0)}$ such that

1. $p\big|_{X^{(0)}} = \mathrm{id}_{X^{(0)}}$,

2. p is constant on open cells in X,

3. $p(x)$ belongs to the smallest subcomplex containing x, for all $x \in X$.

A *morphism* $(X, p) \to (X', p')$ is a skeleton preserving continuous map so that for each $i \in \mathbb{N}$ the diameters of $p'(\sigma)$ are uniformly bounded, where σ are i-cells in $X^{(i)}$.

A *bounded geometry metric cell complex* is a CW complex X equipped with a control map p, whose 1-skeleton $X^{(1)}$ is connected and equipped with a path metric with respect to which all edges have the same length, subject to the condition that there exists a function $D(m)$ so that every closed cell $\sigma \subset X^{(m)}$ intersects at most $D(m)$ closed cells in $X^{(m)}$.

Remark 2.10. Note that for such a complex, the metric space $X^{(0)}$ has bounded geometry in the sense of Section 2.1.

To simplify the terminology, we will refer to bounded geometry metric cell complexes as simply *metric cell complexes*: the bounded geometry will be assumed by default.

We say that a metric cell complex X is *Gromov-hyperbolic* if its zero-skeleton $X^{(0)}$ is Gromov-hyperbolic.

Let X be a metric cell complex. If $V \subset X^{(0)}$ and $R \in \mathbb{Z}_+$, we denote the closed metric R-neighborhood of V in the 0-skeleton by

$$N_R^{(0)}(V) := \{x \in X^{(0)} \mid d(x, V) \leq R\}.$$

Given $m \in \mathbb{Z}_+$, $R \in \mathbb{Z}_+$, and a subcomplex $Y \subset X$, we define the *R-neighborhood of Y in the m-skeleton*, $N_R^{(m)}(Y)$, to be the largest subcomplex of $X^{(m)}$ whose 0-skeleton is $N_R^{(0)}(Y^{(0)})$. Given $x \in X^{(0)}$ we let $B^{(m)}(x, r) := N_R^{(m)}(\{x\})$, be the r-ball in $X^{(m)}$ with the center at x and radius r. Note that

1. $N_0^{(m)}(Y) \supset Y \cap X^{(m)}$,

2. If Y, $Y' \subset X$ are subcomplexes and $R + R' < d(Y^{(0)}, Y'^{(0)})$, then for all $m \in \mathbb{Z}_+$ we have $N_R^{(m)}(Y) \cap N_{R'}^{(m)}(Y') = \emptyset$.

If X is an m-dimensional metric cell complex then we will use the abbreviation $N_R(V) := N_R^{(m)}(V)$.

We will only use the notation $B(x, r)$ to denote closed metric balls in the 1-skeleton $X^{(1)}$, i.e. $B(x, r) := \{y \in X^{(1)} \mid d(x, y) \leq r\}$, where $x \in X^{(1)}$. If Y is a subset of X, we define its diameter to be $\mathrm{diam}(Y) := \mathrm{diam}(p(Y))$. Similarly, we define the distance between two functions $f, f' : S \to X$ to be the quantity $d(p \circ f, p \circ f')$.

A metric cell complex X is called *uniformly k-acyclic* if there is a function $\psi_k(R) = \psi(R)$ such that for each subcomplex $K \subset X^{(k)}$ of diameter $\leq R$ the maps

$$\tilde{H}_i(K) \to \tilde{H}_i(N_{\psi(R)}^{(k+1)} K), i = 0, ..., k,$$

are trivial. A complex X is called *uniformly acyclic* if it is uniformly k-acyclic for each $k = 0, 1, ...$ One defines uniform contractibility in the similar fashion using homotopy groups instead of the homology groups.

An important class of metric cell complexes is given by *Rips complexes* of metric spaces. Let Z be a metric space and $D \in \mathbb{R}_+$. The *D-Rips complex* $\mathrm{Rips}_D(Z)$ is defined to be the simplicial complex whose vertex set is Z, where distinct points $x_0, ..., x_n \in Z$ span an n-simplex in $\mathrm{Rips}_D(Z)$ iff $d(x_i, x_j) \le D$ for all $0 \le i, j \le n$. Thus we get a direct system of Rips complexes $\mathrm{Rips}_D(Z)$ with the inclusion morphisms $\mathrm{Rips}_D(Z) \to \mathrm{Rips}_{D'}(Z)$ for $D \le D'$.

We metrize R-Rips complexes by taking the largest metric for which all simplicial embeddings $\sigma \to \mathrm{Rips}_R(Z)$ are 1-Lipschitz, where σ is a regular Euclidean simplex with side length R. We define the control map $c : \mathrm{Rips}_R(Z) \to Z$ by sending each simplex to one of its vertices.

If Z has bounded geometry, so does $\mathrm{Rips}_R(Z)$ for each $R < \infty$. If Z is δ-hyperbolic then for each $R \ge 10\delta$, the complex $\mathrm{Rips}_R(Z)$ is uniformly contractible, see [11, Lemma 17.A].

Definition 2.11. A metric space Z is said to be *coarsely connected* if there exists $R < \infty$ such that the Rips complex $\mathrm{Rips}_R(Z)$ is connected. Equivalently, Z is coarsely connected if it is quasi-isometric to a geodesic metric space.

Suppose that M is a Riemannian n-manifold whose injectivity radius is bounded from below. Let $\epsilon > 0$ is such that each ϵ-ball in M is convex. Triangulate M so that each simplex is contained in an ϵ-ball in M. Let X denote the Rips complex $\mathrm{Rips}_R(M^{(0)})$ for $0 < R < \epsilon$.

Proposition 2.12. *Under the above assumptions the manifold M is a deformation retract of X.*

Proof. There is a natural simplicial map

$$\iota : M \to X$$

which sends each vertex of M to itself. We construct the retraction $\rho : X \to M$ by the induction on skeleta of X. Each vertex x in X maps to the corresponding point $x \in M^{(0)}$. Suppose that for an i-skeleton $X^{(i)}$ we have constructed a map $\rho : X^{(i)} \to M$ such that the image of each i-simplex $[x_0, ..., x_i]$ is contained in the convex hull of $\{x_0, ..., x_i\}$, which is in turn contained in an ϵ-ball containing $\{x_0, ..., x_i\}$. Then the image of the boundary of each $i + 1$-simplex $\Delta := [x_0, ..., x_i, x_{i+1}]$ is also contained in the convex hull of $\{x_0, ..., x_i, x_{i+1}\}$. Since the latter is contractible, we can extend the map $\rho : \partial\Delta \to M$ to a map $\rho : \Delta \to M$. We thus get a continuous map $\rho : X \to M$ such that $d(\rho \circ \iota, Id) \le \epsilon$. Hence $\rho \circ \iota$ is homotopic to the identity map. $\square$

Proposition 2.13. *Suppose that Y is a metric cell complex such that $Y^{(2)}$ is simply-connected. Then $\mathrm{Rips}_d(Y^{(0)})$ is 1-connected provided that d is sufficiently large.*

Proof. Let d_i denote the supremum of the diameters of the sets $p(\sigma^i)$, where the supremum is taken over all i-cells $\sigma^i \subset Y^{(i)}$. Since Y is connected, it is clear that $\mathrm{Rips}_d(Y^{(0)})$ is connected for each $d \geq d_1$.

Let $d \geq d_1$. Consider a loop $\gamma : S^1 \to \mathrm{Rips}^{(1)}_d(Y^{(0)})$. After homotoping γ if necessary, we may assume that it is a simplicial map with respect to some triangulation $\mathcal{T}$ of S^1. Define a map $\gamma_1 : S^1 \to Y^{(1)}$ as follows. For each vertex v of $\mathcal{T}$, let $\gamma_1(v) \in Y^{(0)} \equiv \mathrm{Rips}^{(0)}_d(Y^{(0)})$ be equal to $\gamma(v)$. For each edge $e = [v_1v_2]$ of $\mathcal{T}$, let $\gamma_1\Big|_e$ be a geodesic in $Y^{(1)}$ between $\gamma_1(v_1)$ and $\gamma_1(v_2)$. There is a natural map $Y^{(1)} \xrightarrow{i_1} \mathrm{Rips}^{(1)}_d(Y^{(0)})$ which takes each $v \in Y^{(0)}$ to the corresponding vertex of $\mathrm{Rips}^{(0)}_d(Y^{(0)})$ and maps each edge of $Y^{(1)}$ at constant speed to the corresponding edge of $\mathrm{Rips}^{(1)}_d(Y^{(0)})$. Let $\gamma_2 := i_1 \circ \gamma_1$.

If $d \geq d_2$ then i_1 can be extended to a map $Y^{(2)} \xrightarrow{i_2} \mathrm{Rips}^{(2)}_d(Y^{(0)})$. This implies that γ_2 is null-homotopic in $\mathrm{Rips}^{(2)}_d(Y^{(0)})$.

On the other hand, we claim that γ_2 is homotopic to γ in $\mathrm{Rips}^{(2)}_d(Y^{(0)})$. To see this, for each edge $e = [v_1v_2]$ of $\mathcal{T}$, let $y_0 = \gamma(v), y_1, ..., y_m = \gamma(w)$ be the vertices of $Y^{(1)}$ on $\gamma_1(e)$ so that $\gamma_2(e)$ is the concatenation of the edges

$$[y_0y_1], ..., [y_{m-1}y_m] \subset \mathrm{Rips}^{(2)}_d(Y^{(0)}).$$

Since $\gamma_1(e)$ is a geodesic between y_0, y_m and $d_{Y^{(1)}}(y_0, y_m) \leq d$, we get:

$$d_{Y^{(1)}}(y_0, y_i) \leq d, i = 1, ..., m-1.$$

Hence each triple of vertices y_0, y_i, y_m spans a 2-simplex Δ_i in $\mathrm{Rips}^{(2)}_d(Y^{(0)})$. Together these simplices define a homotopy between $\gamma(e)$ and $\gamma_2(e)$ (rel. the end-points). Thus the loops γ and γ_2 are homotopic. ☐

2.8. **Finiteness properties of groups.** We recall that a group G is said to have type $\mathbf{F}_n$ ($n = 1, 2, ..., \infty$) if there exists an $(n-1)$-connected n-dimensional cell complex X and a properly discontinuous free action $G \curvearrowright X$ such that $X^{(i)}/G$ is compact for each $i < \infty$.

This notion of finiteness has the following homological generalization (see [4]): A group G is said to be of type FP_n (over a commutative ring $\mathcal{R}$) if there exists a partial resolution of $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules:

$$P_n \to ... \to P_0 \to \mathcal{R} \to 0.$$

The group G is of type FP if there exists a finite resolution

$$0 \to P_n \to ... \to P_0 \to \mathcal{R} \to 0.$$

of $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules.

Proposition 2.14. *Suppose that there exists an $(n-1)$-connected n-dimensional cell complex Y and a discrete (i.e. properly discontinuous) action $G \curvearrowright Y$ such that $Y^{(i)}/G$ is compact for each $i < \infty$. Then G is of type $\mathbf{F}_n$.*

Proof. We note that if the action $G \curvearrowright Y$ were free, then this action would satisfy the properties stated in the definition of a group of type $\mathbf{F}_n$ and there would be nothing to prove. Our goal is to modify Y to make the action free. We do this by induction on skeleta. Let $i = 1$. Since $G \curvearrowright Y^{(1)}$ is cocompact and Y is connected, we conclude that G is finitely-generated. Hence we take $X^{(1)}$ to be a Cayley graph of G. Suppose that $2 \leq i \leq n$ and an $(i-1)$-connected complex $X^{(i)}$ together with a free discrete cocompact action $G \curvearrowright X^{(i)}$ was constructed. We convert X into a metric cell complex by taking a G-equivariant control map $p : X \to X^{(0)}$. Let $x_0 \in X^{(0)}$ be a base-point.

Lemma 2.15. *There are finitely many spherical i-cycles $\sigma_1, ..., \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_i(X^{(i)}, x_0)$, in the sense that the normal closure of the cycles $\{\hat{\sigma}_j : j = 1, ..., k\}$ is $\pi_i(X^{(i)}, x_0)$, where each $\hat{\sigma}_j$ is obtained from σ_j by attaching a "tail" from x_0.*

Proof. Without loss of generality we can assume that $X^{(i)}$ and Y are simplicial complexes. Let $f : X^{(i)} \to Y^{(i)}$ be a G-equivariant continuous map. Consider the embedding

$$X^{(i)} \hookrightarrow Cone(f)$$

where $Cone(f)$ is the mapping cone. Since $Cone(f)$ is i-connected, according to [12, Proof of Lemma 5.8], there are finitely many spherical i-cycles $\sigma_1, ..., \sigma_k$ in $X^{(i)}$ such that their G-orbits normally generate $\pi_i(X^{(i)})$. Here is the brief outline of the construction of σ_j's:

Let $\tau_\alpha : S^i \to Y^{(i)}, \alpha \in \mathbb{N}$, denote the attaching maps of the $(i+1)$-cells in Y, these maps are just simplicial homeomorphic embeddings from the boundary S^i of the standard $(i+1)$-simplex into $Y^{(i)}$. Starting with a G-equivariant projection $Y^{(0)} \to X^{(0)}$ one inductively constructs a (non-equivariant) map $\bar{f} : Y^{(i)} \to X^{(i)}$ so that $f \circ \bar{f} : Y^{(i)} \to Y^{(i+1)}$ is homotopic to the identity inclusion $id : Y^{(i)} \hookrightarrow Y^{(i+1)}$ via a homotopy H whose tracks have "uniformly bounded complexity": The compositions

$$H \circ (\tau_\alpha \times id) : S^i \times I \to Y^{(i+1)}$$

are simplicial maps with a uniform upper bound on the number of simplices in a triangulation of $S^i \times I$. We let σ_α denote the composition $g_\alpha \circ \bar{f} \circ \tau_\alpha$ where $g_\alpha \in G, \alpha = 1, ..., k$ are chosen so that the image of σ_α intersects a fixed fundamental domain for the action $G \curvearrowright X^{(i)}$. □

We now equivariantly attach $(i+1)$-cells along G-orbits of the cycles σ_j: For each j and $g \in G$ we attach an $(i+1)$-cell along $g(\sigma_j)$. Note that if σ_j is stabilized by a subgroup of order $m = m(j)$ in G, then we attach m copies of the $(i+1)$-dimensional cell along σ_j. We let $X^{(i+1)}$ denote the resulting complex and we extend the G-action to $X^{(i+1)}$ in obvious fashion. It is clear that $G \curvearrowright X^{(i+1)}$ is free, discrete and cocompact. $\square$

Corollary 2.16. *Suppose that G is Gromov-hyperbolic. Then G is of type $\mathbf{F}_\infty$.*

Proof. Use the action of G on its contractible Rips complex Y. $\square$

2.9. Gromov's coarse version of the Cartan-Hadamard theorem.

Theorem 2.17. *(Cf.* [11], [2, Theorem 8.1.2]*) There are constants d_0, C_1, C_2, and C_3 with the following property. Let X be a metric space of bounded geometry. Assume that for some δ, and $d \geq \max(C_1\delta, d_0)$, every ball of radius $C_2 d$ in X is δ-hyperbolic, and $\mathrm{Rips}_d(X)$ is 1-connected. Then X is $C_3 d$-hyperbolic.*

One can give a direct proof of this theorem modeled on the proof of the Cartan-Hadamard theorem. Instead of doing this, we will use 6.8M and 6.8N from [11].

Remark 2.18. We note that the δ-hyperbolicity in the statements of 6.8M and 6.8N is to be taken in Gromov's sense.

To prove the above theorem we will need several auxiliary results, which are essentially contained in [11]. The main point of these results is that a (coarsely simply-connected) metric space of bounded geometry is Gromov-hyperbolic iff its Rips complexes satisfy a linear isoperimetric inequality.

Taking $A'_0 = 500d^2$ in [11, 6.8M] we get:

Theorem 2.19 (6.8M, adapted version)**.** *Suppose that X is a metric space of bounded geometry, such that for some $d \geq 0$ every simplicial circle S' in $P^1_d(X)$ with*

$$500d^2 \leq A(S') \leq 64(500d^2)$$

satisfies

$$L(S') \geq d\sqrt{(4000)(64)(500)} \tag{2.20}$$

and $P^2_d(X)$ is 1-connected. Then $P^1_d(X)$ is $(400)\sqrt{500}d$-hyperbolic in the sense of Rips (see [11, 6.8.J]*) and X is $(400)\sqrt{500}d$-hyperbolic in the sense of Gromov.*

Here $P^m_d(X)$ denotes the m-skeleton of the Rips complex $\mathrm{Rips}^{(m)}_d(X)$ (endowed with a metric for which each simplex is path-isometric to a regular Euclidean simplex of side length d). The quantities $L(S')$ and $A(S')$ are the length of S' and the minimal area of a null-homotopy of S', where length and area are computed using the metric on $P^2_d(X)$ rather than the combinatorial length and area.

Theorem 6.8N from [11] states

Theorem 2.21 (6.8N)**.** *If X is δ-hyperbolic and $d \geq 8\delta$, then every simplicial circle $S' \subset P^1_d(X)$ satisfies $L(S') \geq \frac{d}{4\sqrt{3}} A(S')$.*

Proof of Theorem 2.17. Choose d_0 such that

$$\frac{500 d_0^2}{4\sqrt{3}} \geq \sqrt{(4000)(64)(500)}, \tag{2.22}$$

and set $C_1 := 32$, $C_2 := 64 \cdot 500$. Let $S' \subset P^1_d(X)$ be a simplicial circle with

$$500 d^2 \leq A(S') \leq 64(500 d^2) \tag{2.23}$$

and let $f : D \to P^2_d(X)$ be a least area simplicial 2-disk filling S'. There are at most (64)(500) triangles in the triangulated 2-disk D which are mapped isomorphically by f, by (2.23). Therefore, if we look at $\mathrm{Im}(f) \subset P^2_d(X)$, and let $W \subset \mathrm{Im}(f)$ be the closure of the union of 2-simplices contained in $\mathrm{Im}(f)$, then connected components W_i of W have diameter $\leq (64)(500)d$. This means that we can decompose D along disjoint arcs as the union of disks $D_i, i = 1, ..., k+1$ and regions E_j, so that each $f(E_j)$ is at most 1-dimensional and the diameter of each "minimal 2-disk" $f(D_i)$ is at most $(64)(500)d$.

By assumption, every ball of radius $C_2 d = 64 \cdot 500 d$ is δ-hyperbolic and $d \geq C_1\delta = 32\delta$, so by applying Theorem 2.21 to $f(\partial D_1), \dots f(\partial D_{k+1})$ and adding up the results, we obtain

$$L(S') \geq \frac{d}{4\sqrt{3}} A(S') \geq \frac{d}{4\sqrt{3}} 500 d^2 \geq \frac{d}{4\sqrt{3}} 500 d_0^2 \geq d\sqrt{(4000)(64)(500)}$$

where the last inequality comes from (2.22). By Theorem 2.19 we conclude that X is $C_3 d$-hyperbolic (in Gromov's sense) where $C_3 := (400)(\sqrt{500})$. $\square$

As a corollary we get:

Corollary 2.24. *There exist a constant $0 < c < \infty$ such that for each 1-connected 2-dimensional metric cell complex Y, there exists a constant $\rho = \rho(Y)$ with the property:*

If for some $R \geq \rho$, each ball $B_R(y) \subset Y$ is cR-hyperbolic (in the sense of Rips), then Y is Gromov-hyperbolic.

Proof. First of all, since Y is 1-connected, there exists a constant $D = D(Y)$ such that $\mathrm{Rips}^{(2)}_d(Y^{(0)})$ is 1-connected for each $d \geq D$, see Proposition 2.13.

Let C_1, C_2 and d_0 be the constants from Theorem 2.17. Choose ρ so that $\rho/C_2 \geq \max(D, d_0)$. Let $c := \frac{1}{4C_1C_2}$.

Set $d := R/C_2$ and $\delta := cR$. Then $\mathrm{Rips}^{(2)}_d(Y^{(0)})$ is 1-connected. If each ball $B_R(y) \subset Y$ is δ-hyperbolic (in the sense of Rips) then for each $x \in Y^{(0)}$, the ball

$B_R(x) \subset Y^{(0)}$ is 4δ-hyperbolic (in Gromov's sense). Since (by our choice of the constant c)

$$d = \frac{R}{C_2} \geq 4C_1C_2cR$$

(in fact, the equality holds), and

$$d \geq \frac{\rho}{C_2} \geq \max(D, d_0),$$

Theorem 2.17 implies that Y is Gromov-hyperbolic. □

Corollary 2.25. *Suppose that G is a finitely-presented group such that* some *asymptotic cone of G is a tree. Then G is Gromov-hyperbolic; in particular,* every *asymptotic cone of G is a tree. Thus, under the hypothesis of the corollary, all asymptotic cones of G are isometric.*

Proof. Let ω be a nonprincipal ultrafilter on $\mathbb{N}$, let R_j be a sequence of positive real numbers such that $\lim_\omega R_j = \infty$. Let Y be a (simply-connected) Cayley complex for G, $y_j \in Y$ is a sequence of vertices. We give Y structure of a metric cell complex so that G acts on Y isometrically. By our assumption, $\lim_\omega \frac{1}{R_j}(Y, y_j)$ is a tree for some choice of $\omega, (R_j)$ (and the base-points y_j). Thus each geodesic triangle in an R_j-ball $B(y_j, R_j) \subset Y$ is δ_j-thin, where

$$\lim_\omega \frac{\delta_j}{R_j} = 0.$$

Hence the same is true for each ball $B(y, R_j) \subset Y$, $y \in Y^{(0)}$. For sufficiently large j, $R_j \geq \rho = \rho(Y)$ and $\frac{\delta_j}{R_j} < c$, where ρ, c are the constants from the previous corollary. Hence, by Corollary 2.24, the complex Y is Gromov-hyperbolic and therefore G is too. □

We note that the above corollary is false for finitely-generated groups, as there are finitely generated groups G so that some asymptotic cones of G are trees and some are not, see [6], [22]. Under the assumption that the continuum hypothesis (CH) fails, Kramer, Shelah, Trent and Thomas proved in [18] that there are 2^{2^ω} asymptotic cones of uniform lattices in absolutely simple Lie groups of rank ≥ 2. Here ω is the cardinality of $\mathbb{N}$.

Question 2.26. Is it true (without any assumptions on CH) that there are finitely presented groups with non-homeomorphic asymptotic cones?

3. Coarse Poincaré duality

Let $\check{H}^*_c(\cdot)$ denote the compactly supported Čech cohomology with coefficients in a commutative ring $\mathcal{R}$. The (relative) homology and cohomology in this section are also taken with coefficients in $\mathcal{R}$. We first recall the usual Poincaré duality:

Theorem 3.1. *Suppose that X is a metric cell complex homeomorphic to an n-dimensional manifold. Then for each closed subset $W \subset X$ and $k \in \mathbb{Z}$ there is an isomorphism*

$$P_{W,k} : \check{H}^k_c(W) \to H_{n-k}(X, X \setminus W)$$

which is local in the following sense: $\mathrm{Supp}(P_{W,k}(\tau)) \subset N_{D_X}(\mathrm{Supp}(\tau))$ *for each $\tau \in Z^k_c(W)$. The constant D_X does not depend on W and τ. The family $\{P_{W,k}\}$ is compatible with homomorphisms induced by inclusions.*

The *coarse Poincaré duality* is a coarse analogue of the above property; we remind the reader that by convention (see section 2) all our metric cell complexes have bounded geometry. Given a subcomplex $K \subset Y := X^{(m)}$ we let $N_R(K)$ denote the R-neighborhood of K in Y and

$$V_R := \overline{Y \setminus N_R(K)}.$$

Definition 3.2. Let X be a uniformly acyclic metric cell complex. We say that X satisfies *coarse n-dimensional Poincaré duality* if the following holds. For each $m \in \mathbb{N}$ there is a constant[2] $D = D_m \geq 0$ so that if $k \in \mathbb{Z}$ and $m \geq 1 + \max(k, n-k)$, then the metric cell complex $Y := X^{(m)}$ satisfies:

There is a system of homomorphisms $\{P_K\}, \{\bar{P}_K\}$ defined for subcomplexes $K \subset Y$:

$$P_K : H^k_c(N_D(K)) \to H_{n-k}(Y, V_0), \bar{P}_K : H^{n-i}_c(Y, V_0) \to H_i(N_D(K))$$

which are compatible with homomorphisms induced by inclusions, and which determine *approximate isomorphisms*[3] in the sense that the homomorphisms $\alpha, \bar{\alpha}$ and $\beta, \bar{\beta}$ in the following commutative diagrams are zero:

$$\begin{array}{ccccccc}
ker(P_{N_D(K)}) & \to & H^k_c(N_{2D}(K)) & \xrightarrow{P_{N_D(K)}} & H_{n-k}(Y, V_D(K)) & \to & coker(P_{N_D(K)}) \\
\alpha \downarrow & & \downarrow & & \downarrow & & \beta \downarrow \\
ker(P_K) & \to & H^k_c(N_D(K)) & \xrightarrow{P_K} & H_{n-k}(Y, V_0(K)) & \to & coker(P_K),
\end{array}$$

$$\begin{array}{ccccccc}
ker(\bar{P}_{N_D(K)}) & \to & H_i(N_D(K)) & \xrightarrow{\bar{P}_{N_D(K)}} & H^{n-i}_c(Y, V_{2D}) & \to & coker(\bar{P}_{N_D(K)}) \\
\bar{\alpha} \uparrow & & \uparrow & & \uparrow & & \bar{\beta} \uparrow \\
ker(\bar{P}_K) & \to & H_i(K) & \xrightarrow{\bar{P}_K} & H^{n-i}_c(Y, V_D) & \to & coker(\bar{P}_K).
\end{array}$$

The homomorphisms P_K (and $\bar{P}_K$) are required to be local in the following sense: if $[\sigma] \in H^k_c(N_D(K))$ is represented by a cocycle $\sigma \in Z^k_c(N_D(K))$, then $P_K(\sigma)$ can be represented by a relative cycle τ supported in $N_D(\mathrm{Supp}(\sigma))$.

---

[2]If X is finite-dimensional, as in Section 3, we can of course take the constant D independent on m and replace $X^{(m)}$ with X, etc.

[3]Vanishing of $\alpha, \bar{\alpha}$ means the *approximate monomorphism* and vanishing of $\beta, \bar{\beta}$ means the *approximate epimorphism.*

We say that a metric cell complex X satisfies *coarse n-dimensional Poincaré duality in dimension j*, if the constant $D = D_m$ and the family of operators $\{P_K\}, \{\bar{P}_K\}$, as above, exists for $k = j$, and $m = 1 + \max(j, n - j)$ (we do not require X to be uniformly acyclic).

Remark 3.3. For most of this paper and in [14], we only use existence of the approximate isomorphisms P_K. The existence of $\bar{P}_K$ will be used in section 5.

Definition 3.4. A bounded geometry metric space Z satisfies coarse n-dimensional Poincaré duality (over $\mathcal{R}$) if there exists a metric cell complex Y whose 0-skeleton is quasi-isometric to Z, so that Y satisfies coarse n-dimensional Poincaré duality.

Definition 3.5. A 1-connected metric cell complex Y satisfying 2-dimensional coarse Poincare duality in dimension 1 will be called a *quasi-plane* (over $\mathcal{R}$).

Lemma 3.6. *(See* [14]*.) Suppose that $n \geq 2$ and X is a metric cell complex which satisfies coarse n-dimensional Poincaré duality over $\mathcal{R}$ in dimension* 1*. Then $X^{(1)}$ is 1-ended.*

We note that in [12] we have proven a number of coarse versions of Jordan separation theorem for complexes satisfying coarse Poincaré duality. In particular:

Proposition 3.7. *Suppose that Z is a (finite-dimensional) metric cell complex satisfying n-dimensional Poincaré duality, M is a bounded geometry uniformly acyclic cell complex which is homeomorphic to an n-manifold. Then for each uniformly proper map $f : M \to Z$, the image $f(M^{(0)})$ is a net in $Z^{(0)}$.*

We next note that (unlike the usual Poincaré duality) coarse Poincaré duality is a quasi-isometry invariant property:

Proposition 3.8. *(See* [14]*.) Suppose X and X' are (uniformly acyclic) cell complexes with quasi-isometric* 0*-skeleta. Then X satisfies coarse n-dimensional Poincaré duality iff X' does.*

Remark 3.9. The setting in [12] and [14] is that of (finite-dimensional) *metric simplicial complexes.* To adopt the results of [12] and [14] to the discussion in the present paper, one has to (inductively) replace each k-skeleton of a metric cell complex X with an appropriate metric simplicial complex $C^{(k)}$. Since in the formulation of the coarse Poincaré duality for metric cell complexes, the constants D_m are allowed to depend on the dimension, this does not cause problems.

The following theorem is a special case of a more general result proven in [14] for Gromov-hyperbolic metric cell complexes satisfying coarse Poincaré duality:

Theorem 3.10. *(See* [14]*.) 1. Suppose that X is a Gromov-hyperbolic quasi-plane over $\mathcal{R}$. Then $\partial_\infty X$ is homeomorphic to the circle S^1 and $X^{(1)}$ is quasi-isometric to $\mathbb{H}^2$.*

2. Suppose in addition that $G \curvearrowright X$ is a cobounded quasi-action. Then this quasi-action is quasi-isometrically conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.

4. The surrounding function

Let Y be a metric cell complex whose 1-skeleton $Y^{(1)}$ is 1-ended, connected, and equipped with a path metric d where edges have length 1. We suppose further that $Y^{(0)}$ is *quasi-homogeneous*, i.e. there are constants L_0, $A_0 \in \mathbb{R}$ such that for all y, $y' \in Y^{(0)}$, there is an (L_0, A_0)-quasi-isometry $Y^{(0)} \to Y^{(0)}$ which maps y to y'.

Definition 4.1. A subgraph $\Gamma \subset Y^{(1)}$ *surrounds* a subset $\Sigma \subset Y^{(1)}$ if $\Gamma \cap \Sigma = \emptyset$, and Γ separates Σ from infinity, i.e. any proper path $\mathbb{R}_+ \to Y^{(1)}$ starting at Σ intersects Γ.

If $\Gamma \subset Y^{(1)}$ is a subgraph, we let $|\Gamma|$ denote the cardinality of the vertex set of Γ; we will refer to $|\Gamma|$ as the *size* of Γ.

Definition 4.2. Given $y \in Y^{(0)}$, $R \in \mathbb{R}_+$, let $\mathrm{Sur}(y, R)$ be the minimum of sizes of connected subgraphs $\Gamma \subset Y^{(1)}$ which surround $B(y, R) \subset Y^{(1)}$. We will refer to a graph Γ which realizes the minimum as a *smallest* graph which surrounds $B(y, R)$.

Lemma 4.3. *The function* $\mathrm{Sur}(\cdot, \cdot)$ *is "quasi-invariant": There are constants C and R_0 such that* $\mathrm{Sur}(y', \frac{R}{C}) \leq C\,\mathrm{Sur}(y, R)$ *for all y, $y' \in Y$, $R > R_0$.*

Proof. Pick y, $y' \in Y^{(0)}$. There are (L, A)-quasi-isometries $f : Y^{(0)} \to Y^{(0)}$ and $f' : Y^{(0)} \to Y^{(0)}$ such that $f(y) = y'$, $f'(y') = y$, and $d(f' \circ f, \mathrm{id}_{Y^{(0)}}) < A$, $d(f \circ f', \mathrm{id}_{Y^{(0)}}) < A$, where L, A are independent of y, y'.

Choose a smallest connected subgraph $\Gamma \subset Y^{(0)}$ which surrounds $B(y, R)$. Then $d(y', f(\Gamma^{(0)})) \geq \frac{1}{L}R - A$. If $x' \in Y^{(0)}$ and $Ld(y', x') + A \leq R$, then $f'(x') \in B(y, R)$, and consequently if $\rho : \mathbb{R}_+ \to Y^{(1)}$ is any proper path with $\rho(0) = x'$, then by the uniform connectedness of $Y^{(0)}$, the set $f'((\mathrm{Im}(\rho) \cap Y^{(0)})$ must pass within distance $D_1 = D_1(L, A, Y)$ of Γ, which means that ρ must pass within distance $D_2 = D_2(L, A, Y)$ of $f(\Gamma^{(0)})$. Applying the uniform connectedness of $Y^{(1)}$ again, we can find a connected subgraph $\Gamma' \subset Y^{(1)}$ such that

$$N_{D_2}(f(\Gamma^{(0)})) \subset \Gamma' \subset N_{D_3}(f(\Gamma'))$$

for $D_3 = D_3(L, A, Y)$. If R' satisfies $R' + D_3 < \frac{1}{L}R - A$, then $B(y', R') \cap \Gamma' = \emptyset$, and hence Γ' surrounds $B(y', R)$. Since $Y^{(1)}$ has bounded geometry, there is a constant $c = c(Y)$ such that $|\Gamma'| \leq c|f(\Gamma^{(0)})| \leq c|\Gamma|$. Taking $C = \max(c, 2L)$, and R_0 sufficiently large, the lemma follows. ∎

Observe that the quantity $\mathrm{Sur}(y, R)$ is finite since the space $Y^{(1)}$ is 1-ended. Note also that, since Y is quasi-homogeneous, there are constants L, A such that for each point $y \in Y^{(0)}$ there exists a 1-Lipschitz (L, A) quasi-geodesic γ through y (with $L \geq 1, A \geq 0$).

Lemma 4.4. *1. For each $R \geq R_0 := 2AL^2$ and each connected subgraph Γ which surrounds $B(y, R)$ we have:* $\operatorname{diam}(\Gamma) \geq \frac{1}{L^2}R$.

2. For the constant $C_0 = 1/L^2$ and all $R \geq R_0$, we have: $\operatorname{Sur}(y, R) \geq C_0 R$.

3. Suppose that Γ is a graph which surrounds a ball $B(y, R)$, $R \geq R_0$. Then for $r < R/L^2$ and each $z \in \Gamma$ the size of $\Gamma \cap B(z, r)$ is at least r.

4. There is a constant $C_1 = C_1 \geq 1$ such that any connected subgraph Γ which surrounds an R-ball $B(y, R)$ must lie in $B(y, C_1|\Gamma|)$.

Set $C_2 = \max(2C_1, 1/C_0)$.

5. If Γ surrounds $B(y, R) \subset Y^{(1)}$, then the connected component of y in $Y^{(1)} \setminus \Gamma$ is contained in $B(y, C_2|\Gamma|)$.

Proof. 1. Let $\Gamma \subset Y^{(1)}$ be a connected graph which surrounds the ball $B(y, R)$. Consider an (L, A) quasi-geodesic $\gamma : \mathbb{R} \to Y^{(1)}$ as above, $\gamma(0) = y$. Since Γ surrounds $B(y, R)$ there are two points $y_\pm \in \Gamma$ such that $\gamma(T_\pm) = y_\pm$, with $T_- < 0 < T_+$. Thus

$$\operatorname{diam}(\Gamma) \geq d(y_-, y_+) \geq \frac{1}{L}(T_+ - T_-) - A \geq \frac{1}{L}(\frac{2}{L}R - 2A) - A \geq \frac{1}{L^2}R,$$

for $R \geq 2AL^2$. This proves (1).

2. Let Γ be a smallest connected graph surrounding $B(y, R)$, where $R \geq R_0$. Then, since Γ is connected, part (1) implies that

$$\operatorname{Sur}(y, R) = |\Gamma| \geq diam(\Gamma) \geq \frac{1}{L^2}R.$$

3. Recall that, according to (1), $\operatorname{diam}(\Gamma) \geq \frac{1}{L^2}R$. Hence for each $R \geq R_0$ and $r < R/L^2$ and $z \in \Gamma$, the metric sphere $S(z, r) \subset Y^{(1)}$ has nonempty intersection with Γ. Thus, since Γ is connected, the intersection $\Gamma \cap B(z, r)$ contains at least r points, vertices of a path in $\Gamma \cap B(z, r)$ connecting z to $S(z, r)$.

4. Set $C_1 := 2L^2(A + 1)$. Let γ be a quasi-geodesic as in (1). Let's estimate the distance $d(y_\pm, y)$. We have:

$$d(y_\pm, y) \leq L|T_\pm| + A, \quad |T_+ - T_-| \leq L(d(y_+, y_-) + A).$$

Hence

$$d(y_\pm, y) \leq L(L(d(y_+, y_-) + A)) + A = L^2 d(y_+, y_-) + (L^2 + 1)A.$$

However, since Γ is connected, $d(y_+, y_-) \leq |\Gamma|$, and therefore

$$d(y_\pm, y) \leq L^2|\Gamma| + (L^2 + 1)A.$$

If $z \in \Gamma$, then $d(z, y_+) \leq |\Gamma|$, which implies that

$$d(z, y) \leq 2L^2|\Gamma| + (L^2 + 1)A \leq 2L^2(A + 1)|\Gamma|,$$

because $|\Gamma| \geq 1$. Thus $\Gamma \subset B(C_1|\Gamma|, y)$.

5. Consider a point $z \in Y^{(1)} \setminus \Gamma$ which lies in the same component of $Y^{(1)} \setminus \Gamma$ as y. Suppose that $z \notin N_R(\Gamma)$. Then Γ surrounds both $B(y, R)$, $B(z, R)$ and hence, by (4),

$$\Gamma \subset B(y, C_1|\Gamma|), \quad \Gamma \subset B(z, C_1|\Gamma|).$$

By the triangle inequality we conclude that $d(y, z) \leq 2C_1|\Gamma|$. If $d(z, \Gamma) \leq R$ then (by (2)) $d(z, y) \leq R \leq |\Gamma|/C_0$. Therefore $z \in B(y, C_2|\Gamma|)$ in this case as well. □

Lemma 4.5. *Suppose there is a constant $C \geq 2$ such that* $\mathrm{Sur}(y, R) < CR$ *for all $y \in Y^{(0)}$, and all $R \geq 1$. Then $Y^{(0)}$ is* doubling *(see section 2), and hence has polynomial growth.*

Proof. In the proof we will be using the constants C_i and R_0 from the previous lemma.

Without loss of generality we may assume that $C, C_2 \in \mathbb{N}$. Recall that (according to Lemma 4.4, Part 5), if Γ surrounds $B(y, R) \subset Y^{(1)}$ and has size $\leq CR$, then the connected component of y in $Y^{(1)} \setminus \Gamma$ is contained in $B(y, C_2CR)$.

Pick $y \in Y^{(0)}$, $R \geq 1$. Choose a connected graph $\Gamma_1 \subset Y^{(1)}$ with size at most CR which surrounds $B(y, R)$, and set $\mathcal{L}_1 = \{\Gamma_1\}$. Let $\mathcal{N}_1$ be an $\frac{R}{2L^2}$-separated $\frac{R}{2}$-net in Γ_1. Then, by Lemma 4.4 (Part 3), the cardinality of $\mathcal{N}_1$ is at most

$$\frac{|\Gamma_1|}{R/(4L^2)} \leq c := 4L^2C.$$

Let $\mathcal{L}_2$ be a collection of connected subgraphs (each having size at most CR) of $Y^{(1)}$ surrounding the R-balls centered at points in $\mathcal{N}_1$. Proceed inductively in this fashion, building up k layers of surrounding connected subgraphs of $Y^{(1)}$. The union $V_k := \mathcal{N}_0 \cup \ldots \cup \mathcal{N}_k$ has cardinality at most

$$c^{k+1} = (4L^2C)^{k+1}.$$

We claim that the collection of C_2CR-balls centered at points in V_k covers $B(y, \frac{kR}{2})$. To see this, consider a path σ of length at most $\frac{kR}{2}$ starting at y. We inductively break σ into a concatenation of at most k subpaths of length at least $\frac{R}{2}$ as follows. Let σ_1 be the initial segment of σ until it arrives at Γ_1. The path σ_1 terminates within distance $\frac{R}{2}$ of a point $y_1 \in \mathcal{N}_1$. Let σ_2 be the initial segment of $\sigma \setminus \sigma_1$ until it arrives at the connected subgraph $\Gamma_2 \in \mathcal{L}_2$ surrounding $B(y_1, R)$. Et cetera. At each step, the segment σ_i has length at least $\frac{R}{2}$, and (by Lemma 4.4, Part 5) all of them are contained in $\cup_{q \in V_k} B(q, C_2CR)$.

Thus

$$B(y, \frac{kR}{2}) \subset \bigcup_{i=1}^{c^{k+1}} B(y_i, C_2CR),$$

for each $R \geq R_0$, and each $y \in Y^{(0)}$. Choosing k such that $[\frac{k}{2}] = 2C_2C$, and setting $\rho = C_2CR$ we see that

$$B(y, 2\rho) \subset \bigcup_{i=1}^{N} B(y_i, \rho),$$

where $N = c^{4C_2C+2}$ is independent of ρ. Hence $Y^{(0)}$ is doubling and thus has polynomial growth. $\square$

Below is an alternative argument for the polynomial growth of $Y^{(0)}$ in the case when it is quasi-isometric to a (finitely generated) group. Let $V(y, r)$ denote the number of points in $B(y, r) \cap Y^{(0)}$ for $y \in Y^{(0)}$.

Proposition 4.6. *Suppose there is a constant C and a sequence $R_j \in \mathbb{R}_+$ diverging to ∞, such that $\mathrm{Sur}(y, R_j) < CR_j$ for all $y \in Y^{(0)}$, and each j. Assume in addition that $Y^{(1)}$ is the Cayley graph of a finitely generated group Q. Then Q is virtually nilpotent and has at most quadratic growth.*

Proof. We recall (N. Varopoulos, [24]) that if there are constants $C_0, a \in \mathbb{R}$ so that

$$V(y, r) \geq C_0 r^a, \quad \forall r \geq 1, \quad \forall y \in Y^{(0)}, \tag{4.7}$$

then there is $C_1 \in \mathbb{R}$ so that

$$|\partial D| \geq C_1 |D|^{(a-1)/a}$$

for all finite subsets $D \subset Y^{(0)}$. (Here $\partial D = \partial_1 D$ is the "boundary" of D, see section 2.) Let $\mathrm{Inrad}(D)$ denote the radius of the largest metric ball $B(z, r) \subset Y^{(1)}$ such that in $B(z, r) \cap Y^{(0)}$ is contained in D. Hence

$$|\partial D| \geq C' |D|^{(a-1)/a} \geq C_1 [C_0 \cdot \mathrm{Inrad}(D)^a]^{(a-1)/a} = C_2 \,\mathrm{Inrad}(D)^{a-1}.$$

On the other hand, the hypothesis of the proposition implies that for each $y \in Y^{(0)}$ each ball $B(y, R_j) \subset Y^{(1)}$ is surrounded by a connected graph $\Gamma_{R_j} \subset Y^{(1)}$ of the size $\leq CR$ ($R \in \mathbb{N}$). Let D_R denote the vertex set of the component of y in $Y^{(1)} \setminus \Gamma_R$. Then

$$|\partial D_{R_j}| \leq C_3 |D_{R_j}| \leq C_4 R_j \leq C_4 \,\mathrm{Inrad}(D_{R_j})$$

for all $j \in \mathbb{N}$. Thus (4.7) cannot hold in $Y^{(0)}$ for any $a > 2$. Therefore for each $a > 2$ there are sequences $y_j \in Y^{(0)}, r_j \in \mathbb{N}$ so that

$$V(y_j, r_j) \leq C_0 r_j^a.$$

Hence, by the improvement of Gromov's theorem on groups of polynomial growth due to Van den Dries and Wilkie [23], the group Q is virtually nilpotent and moreover its growth $\leq a$ for each $a > 2$. Thus Q has at most quadratic growth. $\square$

We now strengthen our assumptions on Y. We assume in addition that $Y^{(2)}$ is 1-acyclic and satisfies 2-dimensional coarse Poincaré duality in dimension 1, i.e. the statement of Definition 3.2 holds for $n = 2$ and $k = 1 = n - k = m - 1$.

Lemma 4.8. *There are constants D and R_1 such that when $R \geq R_1$, the D-neighborhood of each R-fat geodesic triangle Δ in $Y^{(1)}$ will surround a ball of radius $\frac{R}{10}$.*

Proof. Let $\Delta \subset Y^{(1)}$ be an R-fat geodesic triangle, and let x be a point on one of the sides γ of Δ whose distance to the remaining two sides exceeds R. Let $\alpha \in C^1(\gamma)$ be a 1-cocycle supported in $B^{(1)}(x,1) \cap \gamma$, representing the fundamental class of the side γ relative to its boundary. Construct a map $f : Y^{(2)} \to \gamma$ by letting $f\big|_{Y^{(0)}} \to \gamma$ be a nearest point map, and extending f to $Y^{(2)}$ using the uniform contractibility of γ. For all $y \in Y^{(2)}$ we will have

$$d(p(f(y)), p(y)) \leq d(p(y), \gamma^{(0)}) + C_1 \tag{4.9}$$

where $p : Y \to Y^{(0)}$ is the control map, and C_1 is independent of γ. Using $f\big|_{N^{(2)}_{\frac{R}{3}}(\gamma)}$, pullback α to a 1-cocycle

$$\hat{\alpha} := \left(f\big|_{N^{(2)}_{\frac{R}{3}}(\gamma)} \right)^* (\alpha) \in Z^1(N^{(2)}_{\frac{R}{3}}(\gamma)).$$

Extending $\hat{\alpha}$ by zero, we get a cochain α' in $C^1(N^{(2)}_{\frac{R}{3}}(\Delta))$.

Recall that there is a constant C_2 depending only on $Y^{(2)}$ such that if σ is a 2-cell of $Y^{(2)}$ and τ is a 1-cell appearing in the boundary of σ, then $d(p(\tau), p(\sigma)) < C_2$. Using this and (4.9), it follows that if $R \geq C_3$ for $C_3 = C_3(Y^{(2)})$, σ is a 2-cell in $N^{(2)}_{\frac{R}{3}}(\Delta)$, τ is a 1-cell in the boundary of σ, and $\alpha'(\tau) \neq 0$, then the whole boundary of σ lies outside the $\frac{R}{3}$-neighborhoods of the other two sides of Δ; therefore the boundary of σ lies in $N^{(2)}_{\frac{R}{3}}(\gamma)$, which means that σ itself lies in $N^{(2)}_{\frac{R}{3}}(\gamma)$, and so $\alpha'(\partial\sigma) = \hat{\alpha}(\partial\sigma) = 0$. Thus α' is a 1-cocycle when $R \geq C_3$, which we henceforth assume. The restriction of α' to Δ is nontrivial in H^1 because of excision and the nontriviality of α.

Applying our coarse Poincaré duality assumption we get that α' is "dual" to an element

$$c := P_{N^{(2)}_{\frac{R}{3}}(\Delta)}(\alpha') \in H_1(Y^{(2)}, Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-D}(\Delta))$$

which maps nontrivially to $H_1(Y^{(2)}, Y^{(2)} \setminus N^{(2)}_D(\Delta))$ provided $\frac{R}{3} \geq 2D$, where D is the constant in the statement of coarse Poincaré duality. Since Y is 1-acyclic, this means that the 0-chain ∂c is a linear combination $\partial c = \sum a_i y_i$ with nonzero coefficients, where the y_i's lie outside $N^{(2)}_{\frac{R}{3}-D}(\Delta)$, and some pair y_k, y_l of the support points cannot be joined by a curve in $Y^{(2)} \setminus N^{(2)}_D(\Delta)$.

Set $y_k' := p(y_k)$, $y_l' := p(y_l)$. Then y_k', $y_l' \in Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-r_1}(\Delta)$ for $r_1 = r_1(Y^{(2)})$, provided $\frac{R}{3} \geq r_1$. Also, the points y_k and y_k' (resp. y_l and y_l') lie in the same component of $Y^{(2)} \setminus N^{(2)}_{\frac{R}{3}-r_2}(\Delta)$ for $r_2 = r_2(Y^{(2)}) > r_1$, provided $\frac{R}{3} \geq r_2$. Therefore $N^{(1)}_D(\Delta)$

separates one of the balls $B(y'_k, \frac{R}{10})$, $B(y'_l, \frac{R}{10})$ from infinity when R is sufficiently large. $\square$

Proposition 4.10. *Suppose in addition that $\pi_1(Y)$ is trivial. Then there are constants C_h and R_2 such that if $\mathrm{Sur}(p,r) > C_h r$ for some $p \in Y$, $r \geq R_2$, then Y is Gromov hyperbolic.*

Proof. Here is the intuition behind the proof: If the space Y were not Gromov-hyperbolic, there would be geodesic triangles $\Delta \subset Y$ which surround metric balls of radius $R \gg 1$. Since the surrounding function of Y has sufficiently fast growth, the perimeter of such triangles Δ is much larger than the "inradius" R. This however implies, via a corollary of Gromov's coarse Cartan-Hadamard theorem (Corollary 2.24), that Y is Gromov-hyperbolic, contradicting our hypothesis. Below is the detailed argument.

Let $\rho = \rho(Y^{(2)})$ and c be constants from Corollary 2.24. Let D be the constant as in Lemma 4.8. Let m denote the maximal cardinality of $D+1$-balls in $Y^{(0)}$ (this number is finite by the bounded geometry assumption).

According to Lemma 4.3, there exist $C > 0, R_0 > 0$ so that

$$\mathrm{Sur}(y,r) \leq C\,\mathrm{Sur}(y', Cr) \tag{4.11}$$

for all $y' \in Y^{(0)}$ provided $r > r_0 := R_0/C$, which we will henceforth assume. Pick a number

$$C_h \geq \frac{40mC^2}{c}.$$

We assume that $r \geq \frac{4mC\rho}{C_h}$. Suppose $\mathrm{Sur}(y,r) \geq C_h r$ for some $y \in Y^{(0)}$. Then

$$C_h r \leq \mathrm{Sur}(y,r) \leq C\,\mathrm{Sur}(y', Cr) \tag{4.12}$$

for each $y' \in Y^{(0)}$. Consider a geodesic triangle $\Delta \subset Y^{(1)}$ which is $10Cr$-fat. By Lemma 4.8, there exists $r_1 > 0$ so that if $r \geq r_1/C$, which we will assume from now on, then $N^{(1)}_D(\Delta)$ (the D-neighborhood of Δ in $Y^{(1)}$) surrounds some Cr-ball $B(y', Cr)$. Therefore

$$\frac{C_h r}{C} \leq \mathrm{Sur}(y', Cr) \leq m\,\mathrm{length}(\Delta).$$

Thus such a triangle Δ cannot be contained in a ball of radius $R := \frac{C_h r}{4mC}$.

Therefore, for $\delta := 10Cr$, every triangle contained in a ball of radius R is δ-thin. According to our choice of C_h we have:

$$\frac{\delta}{R} = \frac{10Cr}{C_h r/(4mC)} = \frac{40mC^2}{C_h} = \frac{40mC^2}{C_h} \leq c.$$

Our assumption that $r \geq \frac{4mC\rho}{C_h}$ implies that $R \geq \rho$. Hence (since Y is 1-connected) we are in position to apply Corollary 2.24 and conclude that Y is Gromov-hyperbolic.

The constant R_2 in the Proposition can be defined as

$$R_2 := \max(\frac{4mC\rho}{C_h}, \frac{r_1}{C}, \frac{R_0}{C}).$$

□

Corollary 4.13. *Suppose that Y is a quasi-plane so that $Y^{(0)}$ is quasi-homogeneous. Then either:*

1. The surrounding function of Y is superlinear, i.e. for all C_h there is a constant R_2 such that $\mathrm{Sur}(y, R) > C_h R$ for some $y \in Y^{(0)}$ and all $R \geq R_2$. In this case $Y^{(1)}$ is Gromov-hyperbolic.

2. Or the surrounding function of Y is sublinear, i.e. there is a constant C such that $\mathrm{Sur}(y, R) < CR$ for all $y \in Y^{(0)}$, $R \geq 1$. In this case $Y^{(1)}$ is doubling, and has polynomial growth.

5. Further properties of 2-dimensional coarse Poincaré duality spaces

In this section we will establish certain properties of spaces satisfying 2-dimensional coarse Poincaré duality, which will not be needed in the present paper but will be used in its sequel [13].

Theorem 5.1. *Suppose that X is a metric cell complex satisfying 2-dimensional coarse Poincaré duality, so that $X^{(0)}$ is quasi-homogeneous. Then X is* uniformly linearly acyclic*, i.e. there is a function $R' = R'(i, R)$ which is linear with respect to R such that each cycle $\sigma \in Z_i(X)$ whose support has diameter $\leq R$ bounds a chain $\beta \in C_{i+1}(X)$ whose support is contained in the R'-neighborhood of the support of σ. Moreover, for $i \neq 1$ one can use R' which is independent of R.*

Proof. We first consider the case $i \geq 2$. Let K denote the support of σ. Then, since $2 - i \leq 0$, we have the maps

$$\begin{array}{ccc} H_i(N_{2D}(K)) & \rightarrow & 0 \\ \gamma \uparrow & & \uparrow \\ H_i(N_D(K)) & \rightarrow & H^{2-i}(X, V_{2D}) = 0 \end{array}$$

which form an *approximate monomorphism*, i.e. $\gamma = 0$. Therefore σ bounds a chain within $2D$-neighborhood of K. Hence for $i \geq 2$ we can take $R' = 2D$.

For $i = 0$ the assertion immediately follows from the fact that $X^{(1)}$ is a path metric space.

Lastly, consider $i = 1$. We have the approximate isomorphism:

$$\begin{array}{ccccccc} ker(\bar{P}_{N_D(K)}) & \rightarrow & H_1(N_D(K)) & \xrightarrow{\bar{P}_{N_D(K)}} & H^1_c(X, V_{2D}) & \rightarrow & coker(\bar{P}_{N_D(K)}) \\ \bar{\alpha} \uparrow & & \uparrow & & \uparrow & & \bar{\beta} \uparrow \\ ker(\bar{P}_K) & \rightarrow & H_1(K) & \xrightarrow{\bar{P}_K} & H^1_c(X, V_D) & \rightarrow & coker(\bar{P}_K), \end{array}$$

i.e. $\bar{\alpha} = 0, \bar{\beta} = 0$.

Since X is acyclic, for each R we have an isomorphism

$$H_c^0(V_R) \cong H_c^1(X, V_R).$$

Therefore the unbounded components of V_R do not contribute to $H_c^1(X, V_R)$. Consider a bounded component $C_0 \subset V_0$. For $d = diam(C_0)$ we have

$$C_0 \subset N_d(K)$$

and therefore

$$H_c^1(X, V_0) \to 0 \in H_c^1(X, V_d).$$

Hence we have to estimate d from above. There is a connected subset $K' \subset K$ which surrounds C_0. Therefore, according to Part 1 of Lemma 4.4,

$$R + R_0 \geq \operatorname{diam}(K) + R_0 \geq \operatorname{diam}(K') + R_0 \geq \operatorname{diam}(C_0)/L^2,$$

where $R_0 = 2AL^2$. Thus $d = \operatorname{diam}(C_0) \leq L^2(R + R_0)$ and we can take $R'(R, 1) := L^2(R + R_0)$. $\square$

In the rest of this section we assume that X satisfies coarse 2-dimensional Poincaré duality, the metric space $(Y, d) := X^{(0)}$ is doubling and $G \curvearrowright X$ is a discrete, cocompact quasi-action. Given a subgroup $H \subset G$ we let $H(y) := \{h(y) \mid h \in H\}$ be a quasi-orbit of H in Y. In what follows we will use the metric on $H(y)$ induced from Y.

Our main result is:

Theorem 5.2. *Then*

1. *Every asymptotic cone of Y is homeomorphic to $\mathbb{R}^2$.*

2. *For each subgroup $H \subset G$ there is an integer $k \leq 2$ such that for each $y \in Y$, every asymptotic cone of $H(y) \subset Y$ is homeomorphic to $\mathbb{R}^k$.*

Since Y is doubling, the quasi-orbit $H(y)$ is doubling as well. Therefore, each asymptotic cone of $H(y)$ is a Gromov-Hausdorff limit of a subsequence of $(\lambda_i H(y), y)$, where the sequence of scale factors λ_i converges to zero.

We now examine the asymptotic cones of the quasi-action $G \curvearrowright Y$. Recall that if $f : Y \to Y$ is an (L, A)-quasi-isometry then f induces an L-bi-Lipschitz homeomorphism $f_\eta : Y_\eta \to Y_\eta$ of every asymptotic cone Y_η of (Y, d). Therefore, every asymptotic cone

$$Y_\eta = \lim_\eta (Y, \lambda_i d, z_i^0)$$

of Y yields a uniformly bi-Lipschitz action $G_\eta \curvearrowright Y_\eta$, where G_η consists of maps $Y_\eta \to Y_\eta$ represented by sequences

$$g_i : Y \to Y, g_i \in G, d(g_i(z_i^0), z_i^0) \leq C/\lambda_i.$$

Since $G \curvearrowright Y$ is cocompact, the action $G_\eta \curvearrowright Y_\eta$ is transitive. Since Y is unbounded, it contains a quasi-geodesic ray. Thus Y_η also contains a quasi-geodesic ray and therefore is noncompact.

Proposition 5.3. *Every asymptotic cone of the quasi-action $G \curvearrowright Y$ is a transitive action $G_\eta \curvearrowright Y_\eta$, where the action is topologically conjugate to an isometric action of G_η on the Euclidean plane.*

Proof. First of all, Y is coarsely connected, therefore Y_η is a geodesic metric space. Since Y is doubling, each asymptotic cone Y_η is doubling and therefore locally compact and finite-dimensional, see section 2.6. Therefore, since $G_\eta \curvearrowright Y_\eta$ is transitive and uniformly bi-Lipschitz, it follows from the work of Gleason, Montgomery and Zippin that this action is topologically conjugate to a smooth action on a smooth manifold, $G_\eta \curvearrowright M$, see [20], Sections 6.3 and 4.6 (the latter is needed to handle Property A). We refer the reader to [**?**] for a detailed explanation of how to use the results of [20].

In particular, $M \cong G_\eta/K$, where K is the stabilizer of a point in Y_η. If K were noncompact, the action $G_\eta \curvearrowright Y_\eta$ could not have been uniformly bi-Lipschitz. Thus K is compact and therefore M admits a G_η-invariant Riemannian metric which we fix from now on. Since M is homogeneous, it follows that it has constant scalar curvature.

Lemma 5.4. *Y_η is a 2-dimensional $\mathcal{R}$-acyclic manifold.*

Proof. Suppose that the asymptotic cone Y_η is the ultralimit

$$(Y_\eta, y_\eta) = \lim_\omega (Y, \lambda_i d, y_i^0),$$

where $\lim_\omega \lambda_i = 0$.

We first prove acyclicity of Y_η. Given $r > 0$ consider the r-Rips complex $\mathrm{Rips}_r(Y_\eta)$. Since Y_η is homeomorphic to a homogeneous Riemannian manifold, in view of Proposition 2.12, for each sufficiently small r there is a deformation retraction

$$\mathrm{Rips}_r((Y_\eta)^{(0)}) \to Y_\eta$$

where $(Y_\eta)^{(0)}$ is a certain net in Y_η. Thus it suffices to show that there exists a function $r' = r'(r) \geq r$ such that $\lim_{r\to 0} r'(r) = 0$ and for each sufficiently small r, the inclusion

$$\mathrm{Rips}_r(Y_\eta^{(0)}) \to \mathrm{Rips}_{r'}(Y_\eta^{(0)})$$

induces zero map on the m-dimensional homology groups, $m \geq 1$.

Consider an m-cycle σ_η in $\mathrm{Rips}_r(Y_\omega^{(0)})$,

$$\sigma_\eta = \sum_k a_k \sigma_{k\omega},$$

where $a_k \in \mathcal{R}$, $\sigma_{k\omega}$ are simplices in $\mathrm{Rips}_r(Y_\omega)$. Each simplex $\sigma_{k\omega}$ corresponds to a sequence of simplices σ_{ki} in $\mathrm{Rips}_{r/\lambda_i}(Y)$, $i \in \mathbb{N}$. We also obtain an m-cycle

$$\sigma_i = \sum_k a_k \sigma_{ki}$$

in $\mathrm{Rips}_{r/\lambda_i}(Y)$. Let D_i denote the diameter of the support of σ_i. Since Y is uniformly linearly acyclic (see Theorem 5.1, there exists a constant $C \in \mathbb{R}$ such that each σ_i bounds a simplicial $m+1$-chain

$$\beta_i = \sum_l b_l \tau_{li}, b_l \in \mathcal{R},$$

in $\mathrm{Rips}_{Cr/\lambda_i}(Y)$, so that each vertex of β_i is contained within distance CD_i from the support of σ_i. We now use the fact that the convergence of $(Y, \lambda_i d, y_i^0)$ to Y_η can be taken in Gromov-Hausdorff topology. This gives us $(1 - \frac{1}{i}, \frac{1}{i})$-coarse Lipschitz maps

$$f_i : B(y_i, CD_i) \subset (Y, \lambda_i d, y_i^0) \to (Y_\eta, y_\eta).$$

Taking images of the vertices of β_i under these maps we get chains

$$f_{i*}(\beta_i)$$

in $\mathrm{Rips}_{Cr_i}(Y_\omega)$ where $\lim_\omega r_i = r$. Without loss of generality we may assume that $f_i(\sigma_i) = \sigma_\eta$. By picking sufficiently large i, we obtain $r' = r'(r) := Cr_i$ so that the cycle σ_η bounds the chain $\beta_\eta := f_{i*}(\beta_i)$ in $\mathrm{Rips}_{r'}(Y_\eta)$. This proves $\mathcal{R}$-acyclicity of Y_η.

We now prove that Y_η is 2-dimensional. It suffices to show that $\tilde{H}_m(Y_\eta \setminus \{y_\eta\}) = 0$ for $m \neq 1$. The proof is the same as the argument presented above since each m-cycle σ in Y, $m \neq 1$, bounds an $m+1$-chain β such that

$$\mathrm{Supp}(\beta) \subset N_D(\mathrm{Supp}(\sigma))$$

for certain $D = D(m)$, see the proof of Lemma 5.1. ☐

Corollary 5.5. *Y_η is homeomorphic to $\mathbb{R}^2$.*

Proof. By the previous lemma, Y_η is an $\mathcal{R}$-acyclic noncompact surface; by homogeneity it must be oriented (since a curve with nonzero self-intersection number could not be pulled to infinity), and hence $\mathcal{R}$-acyclicity implies $\mathbb{Z}$-acyclicity. Thus Y_η is homeomorphic to $\mathbb{R}^2$. ☐

We now can finish the proof of Proposition 5.3. Since the action G_ω on Y_η and on M (which is a 2-dimensional Riemannian manifold) is proper and transitive, it follows that M and Y_η are quasi-isometric. Since Y_η is doubling, the manifold M is doubling as well. Therefore M must have curvature equal to zero. This completes the proof of the proposition. ☐

Therefore we have proved the first assertion of Theorem 5.2.

Lemma 5.6. *Suppose $W \subset \mathrm{Isom}(\mathbb{R}^2)$ is a closed subgroup of isometries with an unbounded orbit. Then precisely one of the following holds:*

1. The action $W \curvearrowright \mathbb{R}^2$ is cocompact and the translation subgroup $W_{trans} \subset W$ has rank 2.

2. The translation subgroup W_{trans} has rank 1 (and therefore is isomorphic to $\mathbb{Z}$ or $\mathbb{R}$), each orbit of W_{trans} is contained in a straight line parallel to a 1-dimensional subspace $V \subset \mathbb{R}^2$ and the linear part W_{lin} of W is reducible and preserves V.

Proof. Let $W' \subset W$ be the orientation preserving subgroup; since $[W : W'] \leq 2$ the subgroup W' also has unbounded orbits.

We claim that the translation subgroup W'_{trans} is nontrivial. To see this, pick $g \in W' \setminus \{e\}$. We may assume that g is not a translation, so it is a rotation fixing some point $x \in \mathbb{R}^2$. Taking $h \in W'$ such that $h(x) \neq x$, the elements $g, hgh^{-1} \in W'$ are rotations with distinct fixed points, so $[g, h]$ is a nontrivial translation, and $W'_{trans} \neq \{e\}$ as claimed.

If W'_{trans} has rank 1, then it determines a direction in $\mathbb{R}^2$ which is preserved by the linear part of W, and hence W has finite linear part; it follows that $[W : W'_{trans}] < \infty$, and W does not act cocompactly on $\mathbb{R}^2$ and we are in Case 2.

If W'_{trans} has rank two, its orbit spans $\mathbb{R}^2$, hence $\mathbb{R}^2/W$ is compact and we are in Case 1. $\square$

Let $\mathcal{M}$ be the set of complete Δ-doubling pointed metric spaces together with (L, A)- quasi-actions as in section 2.6. The space $\mathcal{M}$ is compact and we have a continuous $\mathbb{R}$-action on $\mathcal{M}$ given by rescaling the metric (see section 2.6).

Remark 5.7. The subset of $\mathcal{M}$ consisting of points $\zeta = (Z, z, W)$ such that W_{trans} has rank 2 is open.

Recall that we have the Δ-doubling metric space Y and an (L, A)-quasi-action of a group G on Y. The subgroup $H \subset G$ determines a collection of quasi-isometries of Y which we, by abuse of notation, will again denote H. By picking a base-point $y \in Y$ we obtain an element $m = (Y, y, H) \in \mathcal{M}$.

Suppose that $\zeta = (Z, z, W)$ is a point in $\omega(m)$, the ω-limit set of m under the $\mathbb{R}$-action. Then, according to Lemma 5.3, $W \curvearrowright Z$ is topologically conjugate to a group acting properly, isometrically on $\mathbb{R}^2$.

Lemma 5.8. *Suppose that for $m = (Y, y, H)$, the ω-limit set $\Lambda = \omega(m)$ contains a point $\zeta = (Z, z, W)$, where W_{trans} acts transitively on Z. Then $H(y)$ is a net in Y.*

Proof. This follows immediately from Lemma 2.3. $\square$

Lemma 5.9. Λ *is connected.*

Proof. Note that the $\mathbb{R}$-action sends each connected component of Λ to itself. Suppose that $C \subset \Lambda$ is an open and closed subset; it is then $\mathbb{R}$–invariant. On the other hand, the $\mathbb{R}$-action on $\mathcal{M}$ does not increase the distance. Therefore, if $p \in \mathbb{R}(m)$ is such that $d(p, C)$ is strictly less than the distance from C to $\Lambda \setminus C$, then $\mathbb{R}(p)$ cannot accumulate to any point of $\Lambda \setminus C$. $\square$

Suppose now on that W_{trans} does not act transitively on Z. Then the linear part of W is a finite group. It follows that either the order of W_{lin} is at most 12 or $W = W_{lin}$, i.e. W is compact. The latter does not occur for the groups W which appear in $\zeta = (Z, z, W) \in \Lambda$ unless $Wz = z$:

Lemma 5.10. *Suppose that $\zeta = (Z, z, W) \in \Lambda$ and W is compact. Then $Wz = z$.*

Proof. Assume that W does not fix the point z. Then the diameter of the orbit Wz is finite. Define the subset $\Lambda' \subset \Lambda$ to consist of points $\zeta' = (Z', z', W') \in \Lambda$ for which $\mathrm{diam}(W'(z'))$ is finite. On the set Λ' we define the function $d(\zeta') = \mathrm{diam}(W'(z'))$. It is clear that the set Λ' is open in Λ and that the function d is continuous on this set. Moreover, the function d is strictly decreasing under the $\mathbb{R}$-action and

$$\lim_{t\to\infty} d(t(\zeta')) = 0.$$

On the other hand, $d(\zeta) > 0$. This however contradicts the chain-recurrence (see Definition 2.7) of the point ζ under the $\mathbb{R}$-action. $\square$

Observe that the subset Λ' of Λ which consists of ζ for which $d(\zeta) = 0$ is both closed and open. Therefore, Lemma 5.9 implies that if $\Lambda' \neq \emptyset$ then $\Lambda' = \Lambda$. If $\Lambda' = \Lambda$ then Theorem 5.2 follows: Each asymptotic cone of $H(y)$ is a single point.

Therefore, from now on we shall assume that $\Lambda' = \emptyset$, i.e. $W \neq W_{lin}$ for the points $\zeta = (Z, z, W) \in \Lambda$.

Lemma 5.11. *Suppose that $\zeta_i = (Z_i, z_i, W_i)$ is a sequence in Λ which converges to $\zeta = (Z, z, W)$ and the linear part of each W_i is finite. Then the limit of $(Z_i, z_i, W_{i,trans})$ equals (Z, z, W_{trans}).*

Proof. Translations in $\mathbb{R}^2$ can only be approximated by translations or rotations by small angle. On the other hand, since the order of the linear part of W_i is at most 12, the rotation angles are bounded away from zero. $\square$

Lemma 5.12. *Suppose that $\zeta = (Z, z, W) \in \Lambda$ is such that W_{trans} has rank 2. Then W acts transitively on Z.*

Proof. Given ζ as above define $diam(\zeta)$ to be the diameter of the quotient Z/W, which is either 1-dimensional or 2-dimensional metric space. Then $diam$ is a continuous function. This function is strictly decreasing under the action of $\mathbb{R}$:

$$diam(t(\zeta)) = e^{-t} diam(\zeta) < diam(\zeta), t > 0.$$

Therefore, if there exists $\zeta \in \Lambda$ such that $0 < diam(\zeta) < \infty$ we get a contradiction with the chain-recurrence of ζ. $\square$

Corollary 5.13. *The subset Λ'' of Λ which consists of points $\zeta = (Z, z, W)$ such that W_{trans} has rank 2 is both open and closed.*

Proof. By Remark 5.7 the above subset is open. It is closed since limit of a sequence of transitive actions is again transitive. $\square$

Lemma 5.9 shows that if $\Lambda'' \neq \emptyset$ then $\Lambda'' = \Lambda$ and hence, according to Lemma 5.12, W acts transitively on Z for each $(Z, z, W) \in \omega(m)$. Hence Theorem 5.2 follows in this case.

Therefore from now on we can assume that Λ consist of points $\zeta = (Z, z, W)$ such that W_{trans} has rank 1.

Lemma 5.14. *Define the subset $U := \{\zeta = (Z, z, W)\} \subset \Lambda$, which consists of elements ζ for which $W \curvearrowright Z$ has a disconnected orbit. Then U is open.*

Proof. We know that for each $\zeta \in \Lambda$, the action is topologically conjugate to an isometric action corresponding to case 2 of Lemma 5.6, where the translation subgroup is isomorphic to $\mathbb{R}$ or to $\mathbb{Z}$. In the latter case openness is clear. In the former case either all orbits are lines, or precisely one orbit is a line, and all other orbits are a disjoint union of two lines. The latter condition is clearly open in Λ. $\square$

For each $\zeta = (Z, z, W) \in \Lambda'$ we define $dis(\zeta)$ to be the minimal distance from z to a connected component of $W_{trans}(z)$ which does not pass through z; in particular, W_{trans} is connected iff $dis(\zeta) = 0$. It is then clear that dis is a continuous function on $U \subset \Lambda'$.

Lemma 5.15. *For each $\zeta = (Z, z, W) \in \omega(m)$ the orbit $W(z)$ is connected.*

Proof. Suppose that there exists $\zeta = (Z, z, W) \in \Lambda$ such that $dis(\zeta) = \epsilon > 0$. Then, since for each $t > 0$, $dis(t(\zeta)) = e^{-t} \cdot dis(\zeta) < dis(\zeta)$, we obtain a contradiction with chain-recurrence analogously to Lemma 5.12. $\square$

Thus for each $\zeta = (Z, z, W) \in \Lambda$, $W(z)$ is homeomorphic to $\mathbb{R}^k$ where $0 \leq k \leq 2$ and k depends only on m. This concludes the proof of Theorem 5.2. $\square$

6. $PD(2)$ groups are virtually surface groups

In this section, we show how to apply the surrounding function to prove that $PD(2)$ groups over arbitrary commutative rings $\mathcal{R}$ are virtually surface groups. This was proven by B. Eckmann, P. Linnel and H. Müller [7, 8] in the case of $\mathcal{R} = \mathbb{Z}$ and, more recently, by Brian Bowditch for field coefficients, [3]. Our main result is

Theorem 6.1. *Suppose that G is a 2-dimensional Poincaré duality group over a commutative ring $\mathcal{R}$ with a unit. Then G is virtually a surface group.*

6.1. $PD(n)$ **groups over a ring.** Poincaré duality groups of dimension n over rings are generalizations of the fundamental groups of closed aspherical n-manifolds. Thus we begin our discussion with the motivating example of Poincaré duality (over $\mathbb{Z}$) for manifolds. If M is an oriented triangulated (connected) n-manifold, Poincaré duality induces a chain homotopy equivalence $C_c^*(M) \to C_{n-*}(M)$ between the complex of compactly supported simplicial cochains and the simplicial chain complex. If a group G acts freely simplicially on M, then Poincaré duality is G-equivariant, up to twisting by the orientation module. More precisely, let $D := H_c^n(M)$ equipped with the left G-action $g \cdot \alpha := (g^{-1})^* \alpha$. Then Poincaré duality defines a G-equivariant map $C_c^*(M) \to D \otimes_{\mathbb{Z}} C_{n-*}(M)$ where G acts on $D \otimes_{\mathbb{Z}} C_{n-*}(M)$ by the diagonal action. The task of this subsection is to establish a similar statement for Poincaré duality groups in general, with the chain complexes $\tau(A_*)$ and B^* below replacing $D \otimes_{\mathbb{Z}} C_{n-*}(M)$ and $C_c^*(M)$ respectively.

Let $\mathcal{R}$ be a commutative ring with unit. Recall [1], that an n-dimensional Poincaré duality group over $\mathcal{R}$ (for short, $PD(n)$ group over $\mathcal{R}$), is an FP-group over $\mathcal{R}$ (see section 2.8) such that $H^i(G, \mathcal{R}G)$ is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module when $i = n$ and is trivial otherwise. This implies that the cohomological dimension of G over $\mathcal{R}$ is equal to n [4, p. 202], and so we may choose a resolution

$$0 \to A_n \to \ldots \to A_0 \to \mathcal{R} \to 0 \tag{6.2}$$

of the trivial $\mathcal{R}G$-module $\mathcal{R}$ by finitely generated projective $\mathcal{R}G$-modules, [4, p. 199]. Applying the functor $\mathrm{Hom}_{\mathcal{R}G}(\cdot, \mathcal{R}G)$ to the resolution A_*, we obtain cochain complex

$$0 \leftarrow B^n \leftarrow \ldots \leftarrow B^0 \leftarrow 0$$

where $B_i := \mathrm{Hom}_{\mathcal{R}G}(A_i, \mathcal{R}G)$. The complex B_* computes $H^*(G, \mathcal{R}G)$ (recall that $H^n(G, \mathcal{R}G) \cong \mathcal{R}$ and $H^i(G, \mathcal{R}G) = 0$ if $i \neq r$), and hence we can extend B^* to a "dual" resolution

$$0 \leftarrow D \leftarrow B^n \leftarrow \ldots \leftarrow B^0 \leftarrow 0 \tag{6.3}$$

where, following [4, p. 219], we set $D := H^n(G, \mathcal{R}G)$. The $\mathcal{R}G$-bimodule structure on $\mathcal{R}G$ induces a right $\mathcal{R}G$-module structure on the resolution (6.3). We will find it more convenient to convert the right G-action to a left G-action by taking inverses of group elements; in particular, we will view D as a left $\mathcal{R}G$-module.

For each left $\mathcal{R}G$-module M, define a new left $\mathcal{R}G$-module $\tau(M)$, by "twisting" with the G-action on D, i.e. $\tau(M)$ is the $\mathcal{R}$-module $D \otimes_{\mathcal{R}} M$ equipped with the diagonal G-action $g \cdot (d \otimes m) = gd \otimes gm$. Consider the chain complex of left $\mathcal{R}G$-modules obtained by applying the functor τ to the resolution (6.2):

$$0 \to \tau(A_n) \to \ldots \to \tau(A_0) \to \tau(\mathcal{R}) \to 0. \tag{6.4}$$

Lemma 6.5. *1. The $\mathcal{R}G$-chain complex (6.4) is a resolution of $\tau(\mathcal{R})$.*

2. The left G-modules $\tau(\mathcal{R})$ and D are isomorphic.

3. $\tau(A_i)$ is a finitely generated projective $\mathcal{R}G$-module for each i.

Proof. The module D is isomorphic to $\mathcal{R}$ as an $\mathcal{R}$-module, and hence tensoring with D over $\mathcal{R}$ does nothing to $\mathcal{R}$-module structure. So clearly (6.4) is still a resolution by $\mathcal{R}$-modules, and therefore a resolution by $\mathcal{R}G$-modules. This proves 1.

For any $\mathcal{R}G$-module M, the $\mathcal{R}$-module $M \otimes_{\mathcal{R}} \mathcal{R}$ with the diagonal G-action is $\mathcal{R}G$-isomorphic to M, which proves 2.

Before proving 3, we first make a few observations about modules isomorphic to $\mathcal{R}$ and the operation τ.

Let E be an $\mathcal{R}$-module isomorphic to $\mathcal{R}$, and let $\mathcal{R}^\times$ denote the group of units in $\mathcal{R}$. Then the map $\mathcal{R}^\times \to \mathrm{Aut}_{\mathcal{R}}(E)$ which assigns to $r \in \mathcal{R}^\times$ the automorphism $E \to E$ given by $e \mapsto re$, is an isomorphism. Hence if E is an $\mathcal{R}G$-module whose underlying $\mathcal{R}$-module is isomorphic to $\mathcal{R}$, then we obtain a character $\chi : G \to \mathcal{R}^\times$ from the equation

$$g \cdot e = \chi(g)e,$$

$g \in G$, $e \in E$. When we tensor E over $\mathcal{R}$ with an $\mathcal{R}G$-module M and let G act diagonally, then up to $\mathcal{R}G$-isomorphism, we may think of the resulting $\mathcal{R}G$-module $M' = E \otimes_{\mathcal{R}} M$ as having the same underlying $\mathcal{R}$-module as M, where the G-action is twisted by the character χ:

$$(g, m) \mapsto \chi(g)gm,$$

for $g \in G$, $m \in M$. Keeping the above remarks in mind, it is then clear that when we apply τ to the group ring $\mathcal{R}G$, we get back an $\mathcal{R}G$-module isomorphic to $\mathcal{R}G$: To get the isomorphism $\phi : \mathcal{R}G \to \tau(\mathcal{R}G)$ just set $\phi(\delta_g) := \chi(g)\delta_g$ for all $g \in G$. Direct sums are respected by τ, so τ transforms free RG-modules into free $\mathcal{R}G$-modules, and hence projective modules into projective modules. Obviously τ preserves finite generation. Combining these statements yields assertion 3. □

The lemma implies that the resolution (6.3) and the resolution (6.4) both define resolutions of D by left $\mathcal{R}G$-modules. Therefore the two resolutions are $\mathcal{R}G$-chain homotopy equivalent, i.e. there are $\mathcal{R}G$-chain mappings

$$P : B^* \to \tau(A_{n-*}), \quad \bar{P} : \tau(A_*) \to B^{n-*} \tag{6.6}$$

and $\mathcal{R}G$-chain homotopy operators

$$\bar{P} \circ P \overset{H}{\sim} \mathrm{id}_{B^*}, \quad P \circ \bar{P} \overset{\bar{H}}{\sim} \mathrm{id}_{\tau(A_*)} \,. \tag{6.7}$$

6.2. $PD(2)$ **groups and quasiplanes.** The goal of this section is to show that for each $PD(2)$-group G over $\mathcal{R}$ there is a 2-dimensional metric cell complex Y on which G acts freely properly discontinuously and cocompactly so that Y satisfies coarse Poincaré duality in dimensions 0 and 1.

Lemma 6.8. *Suppose that G is an FP_2 group over $\mathcal{R}$. Then there exists a 2-dimensional cell complex Y which is 1-acyclic over $\mathcal{R}$ and a free cocompact cellular action $G \curvearrowright Y$.*

Proof. First we note that since G is FP_1 group over $\mathcal{R}$ and the ring $\mathcal{R}$ is nontrivial, the group G is finitely generated (see [4]).

Consider a Cayley graph Γ of G associated with a finite generating set for G. We let $Y^{(1)} := \Gamma$ be the 1-skeleton of Y. We then have a natural monomorphism of $\mathcal{R}G$-modules:

$$\iota : Z_1(\Gamma) \otimes \mathcal{R} = H_1(\Gamma) \otimes \mathcal{R} \to Z_1(\Gamma, \mathcal{R}) = H_1(\Gamma \otimes \mathcal{R}).$$

Next observe that Γ is homotopy-equivalent to the bouquet of circles B (of course the homotopy-equivalence h is not G-invariant), therefore we have the commutative diagram:

$$\begin{array}{ccc} Z_1(\Gamma, \mathbb{Z}) \otimes \mathcal{R} & \xrightarrow{\iota} & Z_1(\Gamma, \mathcal{R}) \\ h_* \otimes id \downarrow & & h_* \downarrow \\ Z_1(B, \mathbb{Z}) \otimes \mathcal{R} & \xrightarrow{j} & Z_1(B, \mathcal{R}). \end{array}$$

It is clear however that j is onto, hence ι is onto as well. Since G is of type FP_2 over $\mathcal{R}$, the $\mathcal{R}G$-module $H_1(\Gamma, \mathcal{R})$ is finitely generated, see [4, Proposition 4.3, Chapter VIII]. Let $c_1, \ldots, c_k : S^1 \to \Gamma$ denote loops whose images in $H_1(\mathcal{G}, \mathcal{R})$ (under the homomorphism ι) generate $H_1(\Gamma, \mathcal{R})$ as the $\mathcal{R}G$-module. For each $g \in G$ we attach, a 2-disk $D^2_{g,i}$ to Γ along $g \circ c_i$. It is clear that G acts freely on the resulting 2-complex Y and that $H_1(Y, \mathcal{R}) = 0$. □

In what follows we will also need to analyze the case of $PD(2)$-groups which act cocompactly on an $(n-1)$-acyclic (over $\mathcal{R}$) n-dimensional complex Y, $n \geq 2$. Henceforth we assume that Y is such a complex. All (co)homology groups through this section will be with coefficients in $\mathcal{R}$.

It will be convenient to metrize the 1-skeleton of Y so that the edges have unit length, thus Y has a natural structure of a metric cell complex with a control map $Y \to Y^{(0)}$. All (co)homology in the following computations will be taken with coefficients in $\mathcal{R}$.

The cellular chain complex of Y determines a partial resolution of $\mathcal{R}$ by $\mathcal{R}G$-modules:

$$C_n(Y) \to \ldots \to C_1(Y) \to C_0(Y) \to \mathcal{R} \to 0. \tag{6.9}$$

The fundamental lemma of homological algebra, applied to the resolution (6.2) and the partial resolution (6.9), provides chain mappings

$$(f_i : A_i \to C_i(Y))_{0\le i\le n}, \text{ and } (\bar{f}_i : C_i(Y) \to A_i)_{0\le i\le n}$$

and homotopy operators

$$(\bar{f}_i \circ f_i \overset{K_i}{\sim} \mathrm{id}_{A_i})_{0\le i\le n}, \quad (f_i \circ \bar{f}_i \overset{\bar{K}_i}{\sim} \mathrm{id}_{C_i(Y)})_{0\le i<n}.$$

Using f_*, $\bar{f}_*$, K_*, and $\bar{K}_*$, we can transfer the operators P, $\bar{P}$, H, $\bar{H}$ (see (6.6), (6.7)) to $C_*(Y)$ and $C_c^*(Y)$, as chain mappings

$$(P_i : C_c^i(Y) \to \tau(C_{2-i}(Y))_{0\le i\le n}, \quad (\bar{P}_i : \tau(C_{2-i}(Y)) \to C_c^i(Y))_{0\le i\le n}$$

and chain homotopies

$$(H_i : C_c^i(Y) \to C_c^{i-1}(Y))_{0<i\le n}, \quad (\bar{H}_i : \tau(C_i(Y)) \to \tau(C_{i+1}(Y)))_{0\le i<n}.$$

Here we have used the natural isomorphisms of $\mathcal{R}G$-modules

$$\mathrm{Hom}_{\mathcal{R}G}(C_i(Y), \mathcal{R}G) \simeq C_c^i(Y),$$

where as with (6.4), right G-actions are converted to left G-actions. The key fact for us is that $(H_i)_{0<i\le n}$ gives a chain homotopy

$$\bar{P}_i \circ P_i \sim \mathrm{id}_{C_c^i(Y)},$$

and $(\bar{H})_{0\le i<n}$ gives a chain homotopy

$$P_i \circ \bar{P}_i \sim \mathrm{id}_{\tau(C_i(Y))}\,.$$

As there are only finitely many G-orbits of cells in Y, it follows that if σ is an oriented i-cell in Y and $\hat{\sigma} \in C_c^i(Y)$ is the associated element, then

$$P_i(\hat{\sigma}) \in \tau(C_{2-i}(Y)) = D \otimes_{\mathcal{R}} C_{2-i}(Y)$$

can be expressed as a sum $\sum_j d_j \otimes \beta_j$ where the β_j's are oriented $(2-i)$-cells contained in a D_1-neighborhood of σ, where D_1 is a universal constant. Similar statements apply to the operators $\bar{P}_*$, H_*, and $\bar{H}_*$. We therefore deduce that when T is one of these operators and $c \in \mathrm{Domain}(T)$, then $T(c)$ is supported in the D_0-neighborhood of the support of c, where D_0 is independent of c. We now forget about the G-action on Y; henceforth we will identify $\tau(C_*(Y))$ with $C_*(Y)$ and $\tau(C_c^*(Y))$ with $C_c^*(Y)$.

Lemma 6.10. *Y satisfies coarse 2-dimensional Poincaré duality over $\mathcal{R}$ in dimensions $0 \le i < n$.*

Proof. This lemma is a special case of the *Coarse Poincaré Duality Theorem* 6.7 proven in [12] (for the integer coefficients). For the reader's convenience we will outline a proof.

When $K \subset Y$ is a (nonempty) subcomplex we will consider the direct system of tubular neighborhoods $\{N_R(K)\}_{R\geq 0}$ of K and the inverse system of the closures of their complements

$$\{V_R := \overline{Y - N_R(K)}\}_{R\geq 0}.$$

We get four inverse and four direct systems of (co)homology groups with coefficients in $\mathcal{R}$:

$$\{H^i_c(N_R(K))\},\ \{H_i(Y, V_R)\},\ \{H^i_c(Y, N_R(K))\},\ \{H_i(V_R)\}$$

$$\{H^i_c(V_R)\},\ \{H_i(Y, N_R(K))\},\ \{H^i_c(Y, V_R)\},\ \{H_i(N_R(K))\}$$

with the usual restriction and projection homomorphisms. Note that by excision, we have isomorphisms

$$H_i(Y, V_R) \simeq H_i(N_R(K), \partial N_R(K)), \text{ etc.}$$

Extension by zero defines a group homomorphism $C^i_c(N_{R+D_0}(K)) \overset{ext}{\subset} C^i_c(Y)$. When we compose this with

$$C^i_c(Y) \xrightarrow{P} C_{2-i}(Y) \xrightarrow{proj} C_{2-i}(Y, V_R)$$

we get a well-defined induced homomorphism

$$P_{R+D_0} : H^i_c(N_{R+D_0}(K)) \to H_i(Y, V_R).$$

We get, in a similar fashion, homomorphisms

$$H^i_c(N_{R+D_0}(K)) \xrightarrow{P_{R+D_0}} H_{2-i}(Y, V_R) \xrightarrow{\bar{P}_R} H^i_c(N_{R-D_0}(K)) \tag{6.11}$$

$$H^i_c(V_R) \xrightarrow{P_R} H_{2-i}(Y, N_{R+D_0}(K)) \xrightarrow{\bar{P}_{R+D}} H^i_c(Y_{R+2D_0}) \tag{6.12}$$

$$H^i_c(Y, N_{R+D_0}(K)) \xrightarrow{P_{R+D_0}} H_{2-i}(V_R) \xrightarrow{\bar{P}_R} H^i_c(Y, N_{R-D_0}(K)) \tag{6.13}$$

$$H^i_c(Y, V_R) \xrightarrow{P_R} H_{2-i}(N_{R+D_0}(K)) \xrightarrow{\bar{P}_{R+D_0}} H^i_c(Y, V_{R+2D_0}) \tag{6.14}$$

Note that the homomorphisms in (6.11), (6.12), (6.13) and (6.14) inherit the bounded displacement property of P and $\bar{P}$.

We now check that the maps P_R satisfy the approximate monomorphism property stated in Definition 3.2, we leave verification of the rest of the assertions to the reader. Let

$$\xi \in Z^i_c(N_{R+2D_0}(K))$$

be a cocycle representing an element $[\xi] \in Ker(P_{R+2D_0})$, and let $\xi_1 \in C^i_c(Y)$ be the extension of ξ by zero. Then we have

$$P(\xi_1) = \partial\eta + \zeta$$

where $\eta \in C_{2-i}(Y)$ and $\zeta \in C_{2-i}(\overline{Y - N_{R+D}(K)})$. Applying $\bar{P}$ and the chain homotopy H, we get

$$\delta H(\xi_1) + H\delta(\xi_1) = \bar{P} \circ P(\xi_1) - \xi_1 = \bar{P}(\partial\eta + \zeta) - \xi_1$$

so

$$\xi_1 = \delta\bar{P}(\eta) + \bar{P}(\zeta) - \delta H(\xi_1) - H\delta(\xi_1).$$

The second and fourth terms on the right hand side vanish upon projection to $H_c^i(N_R(K))$, so $[\xi] \in Ker(H_c^i(N_{R+2D_0}(K)) \to H_c^i(N_R(K))$. $\square$

Remark 6.15. The 2-complex Y is 1-acyclic and satisfies coarse 2-dimensional Poincare duality in dimension 1. Hence Y is 1-ended by Lemma 3.6.

6.3. **Coarse geometry of quasi-planes.** *Proof of Theorem 6.1.* We prove the theorem by analyzing the coarse geometry of the complex Y along the lines of the sections 4 and 3. The difference however is that (unlike in sections 4 and 3) we do not know that Y is simply-connected; on the other hand, in our present situation Y is quasi-isometric to the finitely-generated group G.

Case 1. G is Gromov-hyperbolic. We defined resolutions $\tau(A_*)$ and B^* as in section 6.1. Since G is Gromov-hyperbolic, it admits a free discrete action on a contractible metric complex Y' so that each skeleton of Y'/G is compact, see Corollary 2.16. Although the complex Y' is not finite-dimensional we get a resolution

$$... \to C_n(Y', \mathcal{R}) \to ... \to C_0(Y', \mathcal{R}) \to \mathcal{R} \to 0$$

by free $\mathcal{R}G$-modules. Thus we get a chain-homotopy equivalence between the above resolution and

$$0 \to \tau(A_2) \to \tau(A_1) \to \tau(A_0) \to \tau(\mathcal{R}) \to 0.$$

Using this chain-homotopy-equivalence we can transfer (as in section 6.2) the Poincaré duality operators

$$B^* \to \tau(A_{2-*}), \quad \tau(A_*) \to B^{2-*}$$

defined in section 6.1 to operators

$$\mathrm{Hom}_{\mathcal{R}G}(C_i(Y'), \mathcal{R}G) \xrightarrow{P'} C_{2-i}(Y', \mathcal{R}G), \quad C_i(Y', \mathcal{R}G) \xrightarrow{\bar{P}'} \mathrm{Hom}_{\mathcal{R}G}(C_{2-i}(Y'), \mathcal{R}G)$$

which are homotopy-inverses of each other. We note that the complex

$$\mathrm{Hom}_{\mathcal{R}G}(C_*(Y'), \mathcal{R}G)$$

is nothing but $C_c^*(Y', \mathcal{R})$ (after ignoring the $\mathcal{R}G$-module structure). The complex Y' is quasi-isometric to the finite-dimensional contractible Rips complex X of G. Hence, by choosing continuous quasi-isometries we get maps

$$Y' \xrightarrow{f} X \xrightarrow{\bar{f}} Y'$$

which are uniformly proper homotopy-inverses. Applying these maps to $P', \bar{P}'$ we get duality operators

$$P : C_c^i(X, \mathcal{R}) \to C_{2-i}(X, \mathcal{R}), \quad \bar{P} : C_i(X, \mathcal{R}) \to C_c^{2-i}(X, \mathcal{R})$$

so that the compositions $P \circ \bar{P}$ and $\bar{P} \circ P$ are chain-homotopic to the identity by homotopies with uniformly bounded tracks.

Applying Lemma 6.10 to X we conclude that X is a quasi-plane over $\mathcal{R}$. Thus we are in position to apply Theorem 3.10: The complex X is quasi-isometric to $\mathbb{H}^2$ and the discrete cocompact action $G \curvearrowright X$ is quasi-isometrically conjugate to a discrete cocompact and isometric action of G on $\mathbb{H}^2$. Therefore G is virtually a surface group. This concludes the proof in the Gromov-hyperbolic case.

Case 2. G is not Gromov-hyperbolic. Then, according to Lemma 3.8 in [21], there exist constants $L \geq 1, c > 0$, a sequence $R_j \in \mathbb{R}_+$ diverging to ∞, such that for each $R = R_j$, there exists an L-bilipschitz embedding

$$f = f_R : S_R \to Y^{(1)}.$$

Here S_R is the circle of radius R in $\mathbb{R}^2$. Let S'_R denote $f(S_R)$.

We recall that Y is a metric cell complex which satisfies coarse 2-dimensional Poincaré duality in the dimension 1, let D_0 be the constant which appears in the definition of the coarse 2-dimensional Poincaré duality. $G \curvearrowright Y$ is a free discrete action such that $Y^{(i)}/G$ is compact for each i; in particular, $Y^{(1)}$ is quasi-isometric to G.

Lemma 6.16. *For each $R = R_j \gg 1$, the graph $N_{D_0}^{(1)}(S'_R)$ surrounds a ball of radius $\frac{1}{2}LR$ in $Y^{(1)}$.*

Proof. Our proof is analogous to the proof of Lemma 4.8 and of the coarse Jordan separation theorem [12, Corollary 7.8].

Take $1 < r < \frac{1}{2}(\sqrt{3}LR - 1)$. Define a retraction $\bar{f} : N_r(S'_R) \to S_R$ as follows: For each vertex y in $N_r(S'_R)$ we let $\bar{f}(y) \in S_R$ be a point x such that $f(x) \in S'_R$ is a nearest point to y. Extend f linearly to the 1-skeleton of $N_r(S'_R)$. The inequality $r < \frac{1}{2}(\sqrt{3}LR - 1)$ ensures that the map $\bar{f}$ extends to the 2-skeleton of $N_r(S'_R)$. The same inequality ensures that $\bar{f} \circ f : S_R \to S_R$ is homotopic to the identity.

Let $\alpha \in C^1(S_R, \mathcal{R})$ be a cocycle whose support set has unit diameter and which represents the generator of $H^1(S_R, \mathcal{R})$. Then the homotopy $\bar{f} \circ f \simeq id$ implies that $\alpha' := \bar{f}^*(\alpha)$ is not null-cohomologous in $N_r(S'_R)$. We now apply the coarse Poincaré duality to the cocycle α' and we get a dual relative 1-cycle

$$\sigma := P(\alpha') \in H_1(Y, Y \setminus N_{r-D_0}(S'_R))$$

which maps nontrivially to $H_1(Y, Y \setminus N_{D_0}(S'_R))$ provided that $r \geq 2D_0$. Since Y is 1-acyclic over $\mathcal{R}$, this means that the 0-chain $\partial\sigma$ is a linear combination $\partial\sigma = \sum a_i y_i$ with nonzero coefficients $a_i \in \mathbb{R}$, where the y_i's lie outside $N_{r-D_0}(S'_R)$, and $\sum a_i y_i$

represents a nontrivial class in $\tilde{H}_0(Y \setminus N_{D_0}(S'_R), \mathcal{R})$. Thus there is a pair of points y_k, y_l in the support of $\sum a_i y_i$, which cannot be joined by a curve in $Y \setminus N_{D_0}(S'_R)$.

It follows that the connected graph $N^{(1)}_{D_0}(S'_R)$ separates one of the balls $B(y_k, r - 2D_0)$, $B(y_l, r - 2D_0)$ from infinity. □

Combining the above lemma with Proposition 4.6 we get:

Corollary 6.17. *The group G is virtually nilpotent and has at most quadratic growth.*

The above corollary together with the fact that G is one-ended implies that G contains a nilpotent subgroup Γ of finite index so that Γ fits into a short exact sequence

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z} \to 1;$$

since Γ is nilpotent, this sequence splits and hence $\Gamma \cong \mathbb{Z}^2$. This concludes the proof of Theorem 6.1. □

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Michael Kapovich:

Department of Mathematics
University of California
Davis, CA 95616
kapovich@math.ucdavis.edu

Bruce Kleiner:

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
bkleiner@math.lsa.umich.edu