ON QUASIHOMOMORPHISMS WITH NONCOMMUTATIVE TARGETS

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Abstract. We describe structure of quasihomomorphisms from arbitrary groups to discrete groups. We show that all quasihomomorphisms are "constructible", i.e., are obtained via certain natural operations from homomorphisms to some groups and quasihomomorphisms to abelian groups. We illustrate this theorem by describing quasihomomorphisms to certain classes of groups. For instance, every unbounded quasihomomorphism to a torsion-free hyperbolic group H is either a homomorphism to a subgroup of H or is a quasihomomorphism to an infinite cyclic subgroup of H.

1. Introduction

Let G be a group and H be a group equipped with a proper left-invariant metric d (e.g., a finitely generated group, equipped with a word metric). A map $f : G \to H$ is called a *quasihomomorphism* if there exists a constant C such that

$$d(f(xy), f(x)f(y)) \le C$$

for all $x, y \in G$. In the case when H is discrete (and in this paper we limit ourselves only to this class of groups, except, briefly, in section 9), f is a quasihomomorphism if and only if the set of *defects* of f

$$D(f) = \{f(y)^{-1}f(x)^{-1}f(xy) : x, y \in G\}$$

is finite. A quasihomomorphism with values in $\mathbb{Z}$ (or $\mathbb{R}$, equipped with the standard metric) is called a *quasimorphism*.

The concept of quasihomomorphisms goes back to S. Ulam [33, Chapter 6], who asked if they are close to group homomorphisms. There is a substantial literature on constructing *exotic* quasimorphisms, i.e., ones which are not close to homomorphisms, going back to the work of R. Brooks [5], see e.g. [9] and references therein; we will refer to quasimorphisms constructed via this procedure as *Brooks quasimorphisms*. On the other hand, very little is known about quasihomomorphisms with values in noncommutative groups. The first Ulam-stability theorem was proven by Kazhdan [23], namely, that ϵ-quasihomomorphisms from amenable groups into the group of unitary transformations of any Hilbert space are ϵ'-close to homomorphisms (with $\lim_{\epsilon\to 0} \epsilon' = 0$). It was proven by Shtern [32] (among other things) that any quasihomomorphism from an amenable group G into $GL(n, \mathbb{R})$ is a bounded perturbation of a homomorphism. Ozawa [28] proved that lattices in $SL(n, K)$ ($n \ge 3$, K is a local field) do not admit unbounded quasihomomorphisms to hyperbolic groups. On the negative side, Burger, Ozawa and Thom proved in [8] that every group containing a free nonabelian subgroup, is not *Ulam-stable*, in the sense of Kahzdan's paper. Rolli

Date: February 13, 2016.

[31] constructed exotic quasihomomorphisms of free groups into groups admitting bi-invariant metrics. After this paper was written, Danny Calegari shared with us an email from Bill Thurston, who noted that"About quasimorphisms to non-abelian groups: they may be hard to construct in general, but it looks like the Heisenberg group will be one interesting case." In the same email Thurston outlined a construction of exotic quasihomomorphisms from hyperbolic 3-manifold groups into the 3-dimensional Heisenberg groups using contact structures on 3-manifolds, although filling in details requires some work; for instance, it is far from clear why quasihomomorphisms defined by Thurston are not close to homomorphisms. It follows from our main result that, for this to be the case, at the very least, one has to assume that the 3-manifold M in Thurston's construction satisfies $b_2(M) \geq 2$. A construction of quasihomomorphisms (not close to homomorphisms) from arbitrary hyperbolic groups to Heisenberg groups, which works in greater generality, but is purely algebraic and avoids contact structures, is presented in our Example 2.11.

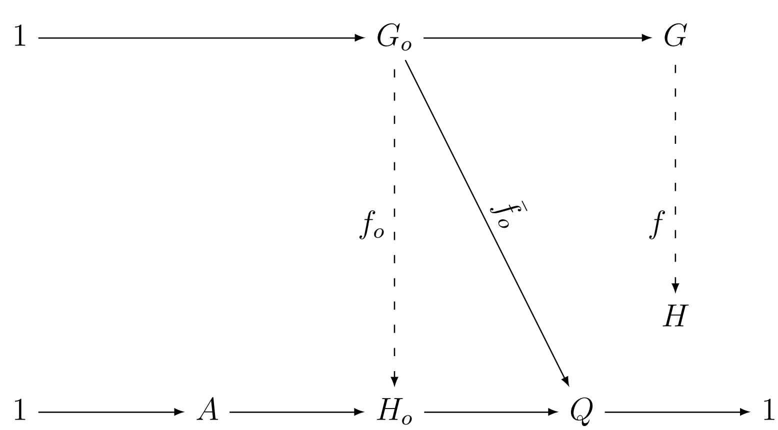
Calegari also brought the paper [10] to our attention, where a certain non-commutative version of quasimorphisms into $\mathbb{R}$ is discussed. Furthermore, after this paper was written we received a preprint by Hartnick and Schweitzer [18], where they proved existence of exotic quasihomomorphisms of free groups; however, their definition of quasihomomorphisms is different from Ulam's. We will discuss their work in more detail in section 9, together with few other generalizations of homomorphisms. In that section we also show that, while Brooks' construction does not generalize to self-quasihomomorphisms of free groups, it does go through when we replace Ulam's notion of a quasihomomorphism with the one of a *middle-quasihomomorphism*.

The goal of this paper is to explain why it is so "hard to construct" quasihomomorphisms to noncommutative groups which are neither homomorphisms, nor come from quasihomomorphisms with commutative targets, provided that H is a discrete group.

In order to formulate our main theorem we will need a definition:

Definition 1.1. A quasihomomorphism $f : G \to H$ is *constructible* (from group homomorphisms) if there exists a finite-index subgroup $G_o < G$, a subgroup $H_o < H$, a finitely generated abelian subgroup $A < H_o$ central in H_o, and a quasihomomorphism $f_o : G_o \to H_o$ within finite distance from $f|G_o$ such that:

The projection $\bar{f}_o : G_o \to Q = H_o/A$ of f_o is a homomorphism.



Special subclasses of quasihomomorphisms include:

1. *Almost homomorphisms*, i.e., maps $f : G \to H' < H$, where H' contains a finite normal subgroup K such that the projection of f to H'/K is a homomorphism.

2. Products of quasimorphisms: $f : G \to H' \cong \mathbb{Z}^n < H$; in this case $f = (f_1, \ldots, f_n)$, where each $f_i : G \to \mathbb{Z}$ is a quasimorphism.

When we cannot specify the quotient group Q in Definition 1.1, but can only claim that it belongs to a certain class $\mathcal{C}$ of groups, we will say that the quasihomomorphism f in this definition is *constructible from quasihomomorphisms to groups in the class* $\mathcal{C}$.

Our main theorem is:

Theorem 1.2. Every quasihomomorphism $f : G \to H$ is *constructible*.

We will prove this theorem in section 3 (see Theorem 3.6).

Remark 1.3. Theorem 1.2 essentially reduces the study of quasihomomorphisms $G \to H$ to analyzing quasihomomorphisms $G_o \to A$, homomorphisms $f' : G_o \to Q$ and cohomology classes $[\omega] \in H^2(Q; A)$ with bounded pull-back classes $f'^*([\omega]) \in H^2(G_o; A)$, see section 2.4.1.

We also show how one can sharpen the main theorem by restricting to special classes of target groups, e.g., some periodic groups (Example 3.3), hyperbolic groups (Theorem 4.1), $CAT(0)$ groups (Theorem 5.5), mapping class groups (Theorem 7.1) and groups acting on simplicial trees (Lemma 8.3). For instance:

1. All quasihomomorphisms to free Burnside groups $B(n, m)$ (with large odd exponent m) are bounded.

2. All unbounded quasihomomorphisms to hyperbolic groups are either almost homomorphisms or have elementary images.

3. All quasihomomorphisms $G \to H = Map(\Sigma)$ to the mapping class group are constructible from homomorphisms to other mapping class groups of surfaces (proper subsurfaces in Σ), see Theorem 7.1 for the more precise statement.

In particular, we will show that higher rank irreducible lattices do not admit unbounded quasihomomorphisms to hyperbolic groups and to mapping class groups. This sharpens the main result of Ozawa in [28], since he could prove it only for lattices in $SL(n, K)$.

Denis Osin [27] extended our results on rigidity of quasihomomorphisms to hyperbolic groups and mapping class groups, to the case of relatively hyperbolic target groups and target groups which act acylindrically on Gromov–hyperbolic spaces. Lastly, we note that Nicolaus Heuer in his thesis [19] studied quasihomomorphisms to Lie groups.

Acknowledgements. We are grateful to the Max-Plank-Institute for Mathematics in Bonn, where this paper was written. We are also grateful to Marc Burger, Danny Calegari, Ursula Hamenstädt, Alessandra Iozzi, Gilbert Levitt, Dan Margalit, Igor Mineyev, Denis Osin, Narutaka Ozawa, Andy Putman, Mark Sapir and Andreas Thom for useful conversations, references and corrections. The first author is supported by Grant-in-Aid for Scientific Research (No. 23244005, 15H05739). The second author was also supported by the NSF grant DMS-12-05312, he is also grateful to the Korea Institute for Advanced Study for its hospitality. We are grateful to the referee for many useful comments and corrections.

2. Preliminaries

In this section we collect some basic facts about quasihomomorphisms.

2.1. **Definition and notation.** Throughout the paper (except for §9), we will be considering quasihomomorphisms to discrete groups, denoted H, equipped with a proper metric d (whose choice will be suppressed in our notation). The reader can think of a finitely generated group equipped with a word metric as the main example. Set $|h| = d(1, h)$.

Definition 2.1. Suppose that a map $f : G \to H$ between groups has the property that $f(G)$ is contained in a subgroup $J < H$, J contains a finite normal subgroup $K \lhd J$, such that the projection $\bar{f} : G \to \bar{J} = J/K$ is a homomorphism. We then will refer to f as an *almost homomorphism*, it is a homomorphism modulo a finite normal subgroup (in the range of f).

Clearly, every almost homomorphism is a quasihomomorphism.

A quasihomomorphism $f : G \to H$ is called *bounded* if its image is a bounded (i.e., finite) subset of H. Note that every map $f : G \to H$ with bounded image is automatically a quasihomomorphism.

A map $f : G \to H$ is a *quasiisomorphism* if it is a quasihomomorphism which admits a *quasiinverse*, i.e., a quasihomomorphism $f' : G \to H$ such that

$$\mathrm{dist}(f' \circ f, id) < \infty, \quad \mathrm{dist}(f \circ f', id) < \infty.$$

Here and in what follows, for maps $f_1, f_2 : X \to Y$ to a metric space (Y, d_Y),

$$\mathrm{dist}(f_1, f_2) = \sup_{x \in X} d_Y(f_1(x), f_2(x)).$$

A quasiisomorphism is *strict* if $f' = f^{-1}$. Two groups G, H are (strictly) quasiisomorphic to each other if there exists a (strict) quasiisomorphism between these groups.

In what follows we will frequently use the notation $\mathcal{N}_R(S) \subset H$ to denote the R-neighborhood of a subset S in a discrete group H equipped with a proper metric. We will also use the notation $h_1 \sim h_2$ for elements $h_1, h_2 \in H$ to denote that

$$d(h_1, h_2) \le Const$$

where $Const$ is a certain uniform constant (which is not fixed in advance). Instead of the notation $\sim$, we will also write write

$$p \sim_S q$$

if $p = qs$ with $p, q \in H, s \in S$ (where the subset S is bounded). For example, for a quasihomomorphism $f : G \to H$ with $D = D(f)$, by the definition,

$$f(ab) \sim_D f(a)f(b)$$

for $a, b \in G$.

For two quasihomomorphisms $f_i : G_i \to H, i = 1, 2$, the notation $f_1 \sim f_2$ will mean that the domain of f_1 is a finite index subgroup $G_1 < G_2$ and that

$$\mathrm{dist}(f_1, f_2 | G_1) < \infty.$$

For a subset D of a group H and $n \geq 2$ we will use the notation D^n to denote the subset of H consisting of products of at most n elements of D. More generally, for two subsets $A, B \subset H$ we let

$$A \cdot B = \{ab : a \in A, b \in B\}.$$

We will use the notation D^{-1} for the set of inverses of elements of D. Then

$$h \sim_D h' \iff h' \sim_{D^{-1}} h.$$

For an element $h \in H$ we let $ad(h)$ denote the inner automorphism of H defined by conjugation via h:

$$ad(h)(x) = hxh^{-1}.$$

The map $ad : H \to Inn(H) < Aut(H)$ is a homomorphism; its image $Inn(H)$ is the group of *inner automorphisms* of H. The quotient group $Out(H) = Aut(H)/Inn(H)$ is the *outer automorphism group* of H.

Given a group H and its subgroup A we let $N_H(A)$ and $Z_H(A)$ denote the normalizer and the centralizer of A in H respectively. For a subgroup $B < H$ we will also use the notation

$$N_B(A) := N_H(A) \cap B, \quad Z_B(A) := Z_H(A) \cap B.$$

2.2. **Elementary properties of quasihomomorphisms.**

Composition. The composition of quasihomomorphisms is again a quasihomomorphism:

$$D(f_2 \circ f_1) \subset D(f_2) \cdot f_2(D(f_1)) \cdot D(f_2).$$

In particular, if f_2 is a homomorphism and $f_2(D(f_1)) = \{1\}$, then $f_2 \circ f_1$ is a homomorphism.

Product construction. Let $f_i : G \to H_i, i = 1, ..., n$ be quasihomomorphisms. Then their product

$$f = (f_1, \dots, f_n) : G \to H_1 \times ... \times H_n$$

is again a quasihomomorphism. Conversely, given a quasihomomorphism

$$f = (f_1, ..., f_n) : G \to H_1 \times ... \times H_n,$$

in view of the composition property above, each component f_i is again a quasihomomorphism.

Closeness of $f(G)$ and $f(G)^{-1}$. Suppose that

$$f : G \to H$$

is a quasihomomorphism. Then for $D = D(f)$ we obtain:

$$\epsilon = f(1) = f(1)f(1)s, \quad s \in D$$

and, hence,

$$\epsilon = s^{-1} \in D^{-1}.$$

Furthermore, for $x \in G$

$$1 = f(xx^{-1})\epsilon^{-1} = f(x)f(x^{-1})s\epsilon^{-1}, s \in D$$

which implies that

$$(f(x))^{-1} = f(x^{-1})s', \quad s' \in D^2. \tag{1}$$

In particular, the sets $f(G), (f(G))^{-1}$ are Hausdorff-close to each other.

2.3. **Quasiaction and bounded displacement property.** By the definition of a quasihomomorphism, for $D = D(f)$:

$$f(xyz) \sim_D f(xy)f(z)$$

and

$$f(xyz) \sim_D f(x)f(yz) \sim_D f(x)f(y)f(z).$$

In particular,

$$f(xy)f(z) \sim_{D^{-1}} f(xyz) \sim_{D^2} f(x)f(y)f(z)$$

and, hence,

$$d(f(xy)h, f(x)f(y)h) \le C_3, \quad \forall h \in f(G), \quad C_3 = \max\{|s| : s \in D^2D^{-1}\}. \tag{2}$$

More precisely,

$$f(xy)h \sim_{D^2D^{-1}} f(x)f(y)h, \quad h \in f(G), x, y \in G. \tag{3}$$

Therefore, the left multiplication by $f(x)$ defines a *quasiaction* of G on $f(G)$. The set $f(G)$ is not literally preserved by this quasiaction, but

$$d(f(x)f(G), f(G)) \le C_1, \quad C_1 = \max\{|s|, s \in D\},$$

for all $x \in G$: For $h = f(y) \in f(G)$,

$$f(x)h \sim_{D^{-1}} f(xy) \in f(G).$$

In view of (2), the defect set $D(f)$ has the property that every element $h \in D(f)$ quasiacts on $f(G)$ with bounded displacement. We define the *defect subgroup* $\Delta = \Delta_f$ of f to be the subgroup of H generated by $D(f)$. It is then immediate that every element of Δ_f (quasi)acts on $f(G)$ with bounded displacement. Equation (3) shows that there exists a finite subset $D' = D'(f) = D^2D^{-1} \subset \Delta_f$ such that for every $s \in D = D(f)$,

$$sh = hs', \quad s' \in D'. \tag{4}$$

Remark 2.2. To verify (4), let $h \in f(G)$ and $s \in D = D(f)$, then

$$h^{-1}sh = f(c)^{-1}f(b)^{-1}f(a)^{-1}f(ab)f(c) \sim_{D^2D^{-1}} f(c)^{-1}f(b)^{-1}f(a)^{-1}f(a)f(b)f(c) = 1$$

where $f(c) = h$ and

$$f(b)^{-1}f(a)^{-1}f(ab) = s.$$

In particular,

$$h^{-1}\Delta_f h \subset \Delta_f. \tag{5}$$

Since for every $h \in f(G)$, $h^{-1} \in f(G)D^2 \subset f(G)\Delta_f$ (see equation (1)), we conclude that

$$h\Delta_f h^{-1} \subset \Delta_f \tag{6}$$

as well. Thus:

Lemma 2.3. The sets $f(G)$ and $f(G)^{-1}$ are contained in $N_H(\Delta_f)$, the normalizer of Δ_f in H. In particular, we obtain a homomorphism

$$G \to N_H(\Delta_f)/\Delta_f$$

whose image is $\langle f(G)\rangle/\Delta_f$.

Let $f : G \to H$ be a quasihomomorphism with the defect subgroup Δ_f. As we just proved, the image of f is contained in $N = N_H(\Delta_f)$. It follows that there is no harm in replacing the group H with the group $\langle f(G)\rangle$. We assume from now on that $H = N = \langle f(G)\rangle$; we continue to work with the restriction of the original left-invariant metric from the target group of f to $\langle f(G)\rangle$.

Remark 2.4. We observe that if the group G is finitely generated, so is the group $\langle f(G)\rangle$: It is generated by $f(S)$ and $D(f)$, where S is a finite generating set of G.

By Lemma 2.3, we also obtain a homomorphism

$$\varphi = \varphi_f : G \to Out(\Delta_f) = Aut(\Delta_f)/Inn(\Delta_f) \tag{7}$$

given by sending $g \in G$ first to the conjugation automorphism

$$\tilde\varphi(g) = ad(f(g)) \in Aut(\Delta_f)$$
$$\tilde\varphi(g)(\delta) = f(g)\delta f(g)^{-1}, \quad \delta \in \Delta_f$$

and then projecting to the group of outer automorphisms. (The quasihomomorphism $\tilde\varphi$, of course, in general, is not a homomorphism.) Similarly, by the same lemma, we have an *antihomomorphism*

$$\psi : G \to Out(\Delta_f),$$

$\psi(g)$ defined by sending g to $\tilde\psi(g) = ad(f(g)^{-1})$ and then projecting to $Out(\Delta_f)$. In view of (1), we have

$$\psi(g) = \varphi(g^{-1}).$$

Since Δ_f is generated by the finite subset $D(f)$, the automorphisms $\tilde\varphi(g), \tilde\psi(g)$ are determined by their values on the elements $s \in D(f)$; the images of elements $s \in D(f)$ under $\tilde\varphi(g)$ and $\tilde\psi(g)$ belong to a finite subset $D'(f)$ (independent of g). Therefore, the subset

$$\tilde\varphi(G) \cup \tilde\psi(G) \subset Aut(\Delta_f)$$

is finite and, thus, the homomorphism φ has finite image. We summarize these simple (but useful) observations in

Lemma 2.5. 1. There exists a finite subset $\{y_1, ..., y_n\}$ of H such that

$$\tilde\varphi(G) \cup \tilde\psi(G) \subset \{ad(y_j) : j = 1, ..., n\}.$$

2. The kernel $G_o = \ker(\varphi)$ is a subgroup of finite index in G. For every $g \in G_o$ the automorphisms $\tilde\varphi(g), \tilde\psi(g) \in Aut(\Delta_f)$ are inner. In particular, we can choose the elements $y_1, ..., y_n \in \Delta_f$ such that

$$\tilde\varphi(G_o) \cup \tilde\psi(G_o) \subset \{ad(y_j) : j = 1, ..., n\}.$$

2.4. **Lift and projection.**

2.4.1. *Quasisplit exact sequences.* Consider an exact sequence

$$1 \to A \stackrel{i}{\to} B \stackrel{p}{\to} C \to 1. \tag{8}$$

In what follows, we will identify A with $i(A)$. If A is central in B, then the sequence (8) defines a central extension of C by A.

A sequence (8) is said to be *quasisplit* if there exists a quasihomomorphism $C \stackrel{s}{\to} B$ such that $p \circ s = id$. (More generally, one can allow this composition to have bounded displacement, but we will not need this.) Given a quasisplitting s we define the mapping

$$q(b) = b^{-1} \cdot (s \circ p(b)), \quad q : B \to A.$$

Lemma 2.6. If A is central in B, the map q is a quasihomomorphism.

Proof. Pick $b_1, b_2 \in B$ and set $c_i = p(b_i)$,

$$s(c_i) = a_i b_i, \quad a_i = q(b_i) \in A, \quad i = 1, 2.$$

Then

$$s(c_1 c_2) = s(c_1) s(c_2) \delta, \quad \delta \in D(s).$$

Then,

$$q(b_1 b_2) = b_2^{-1} b_1^{-1} \cdot s(c_1 c_2) = b_2^{-1} b_1^{-1} s(c_1) s(c_2) \delta =$$
$$b_2^{-1} a_1 s(c_2) \delta = a_1 b_2^{-1} s(c_2) \delta = a_1 a_2 \delta = q(b_1) q(b_2) \delta. \quad \square$$

We continue with the hypothesis of the lemma and define the maps

$$F : B \to C \times A, \quad F(b) = (p(b), q(b))$$

and

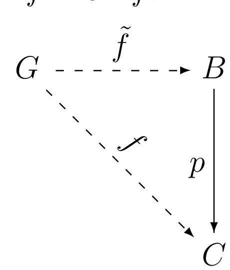
$$F' : C \times A \to B, \quad F'(c, a) = s(c) a^{-1}.$$

Since p and q are (quasi)homomorphisms, so is F. The proof that F' is a quasihomomorphism is completely analogous to the proof of Lemma 2.6 and is left to the reader.

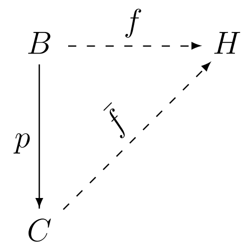
Lemma 2.7. If A is central in B then $F' = F^{-1}$; in particular, the group B is strictly quasiisomorphic to $C \times A$.

Proof. $F' \circ F(b) = F'(p(b), q(b)) = sp(b) \cdot (q(b))^{-1} = sp(b) \cdot sp(b)^{-1} \cdot b = b$. The reader will verify that $F \circ F' = id$. $\square$

Given a quasisplit extension (8), each quasihomomorphism $f : G \to C$ *lifts* to a quasihomomorphism $\tilde{f} : G \to B$, $\quad \tilde{f} = s \circ f$.



Similarly, given a quasisplit exact sequence (8), each quasihomomorphism $f : B \to H$ *projects* to a quasihomomorphism $\bar{f} = f \circ s : C \to H$.



If $f : G \to C$ is unbounded, the quasihomomorphism $\tilde{f}$ is, of course, unbounded as well. This is not necessarily the case for projections of quasihomomorphisms $C \xrightarrow{f} H$ as one can take, for instance, $B = A \times C$ and $f = f_1 \times f_2 : G \to B$, with bounded f_2 and unbounded f_1. However, if A is finite and f is unbounded, then $\bar{f}$ is unbounded as well. We will use this observation several times in the case when $H = \mathbb{Z}$, in order to construct unbounded quasimorphisms on the quotient group C.

Example 2.8. Examples of quasisplit sequences are given by:

a. Extensions with finite kernel A: In this case *any* section $s : C \to B$ will define a quasisplitting.

b. Central extensions whose obstruction class is a bounded 2nd cohomology class, cf. [16] or [25].

The first example is immediate. To justify (b), suppose that $\omega \in Z^2(C, A)$ is a bounded *normalized* cocycle, i.e., $\omega(1, c) = \omega(c, 1) = 0 \in A$ for all $c \in C$. Here and in what follows we use the restriction of the metric from B to $i(A) \cong A$. We also refer the reader to [9] for the discussion of bounded cohomology.

Following [6, p. 92], we define the extension E_ω of C by A, using the group law on the product $A \times C$ given by the formula:

$$(a_1, c_1)(a_2, c_2) = (a_1 + a_2 + \omega(c_1, c_2), c_1 c_2).$$

The group E_ω is then a central extension of C by A, which is isomorphic to the one in (8). The quasisplitting of the sequence

$$0 \to A \to E_\omega \to C \to 1$$

is given by $s(c) = (0, c)$. Then ω is bounded if and only if s is a quasihomomorphism. We obtain

Lemma 2.9. A central extension (8) quasisplits if and only if the extension class is bounded.

In §6 we will prove Proposition 6.4 about quasisplitting of a central extension associated with a certain subgroup of the mapping class group of a surface, illustrating this result.

2.4.2. *Second bounded cohomology of G.* Note that there are situations when the sequence (8) does not quasisplit, but *homomorphisms* $f : G \to C$ still lift to quasihomomorphisms $\tilde{f} : G \to B$. Namely, assume that the subgroup A is central in B and the class $f^*([\omega]) \in H^2(G; A)$ is bounded. Then the homomorphism f lifts to a quasihomomorphism $\tilde{f} : G \to B$. To see this, consider the central extension of G by A defined by the class $f^*([\omega])$:

$$0 \to A \to \tilde{E} \to G \to 1.$$

Let $\tilde{s} : G \to \tilde{E}$ be the quasisplitting. Composing $\tilde{s}$ with the natural homomorphism $\hat{f} : \tilde{E} \to B$ (which projects to $f : G \to C$), we obtain the required lift $\tilde{f}$. The converse to this is also easy to see: If f lifts to a quasihomomorphism $\tilde{f}$, then the class $f^*([\omega]) \in H^2(G; A)$ is bounded.

Example 2.10. Consider the case where A is a finitely generated abelian group central in $\tilde{E}$ and the group G is hyperbolic. Then all cohomology classes in $H^2(G; A)$ are bounded (see [25]), which implies that quasihomomorphisms $f : G \to C$ always lift to quasihomomorphisms $G \to B$.

Example 2.11. Consider the integer Heisenberg group $B = H_{2n}$, where $A \cong \mathbb{Z}$, $C \cong \mathbb{Z}^{2n}$ and the obstruction class $[\omega]$ is unbounded (the cocycle ω is the restriction of a symplectic form from $\mathbb{R}^{2n}$ to $\mathbb{Z}^{2n}$). Then every homomorphism $f : G \to \mathbb{Z}^{2n}$ from a hyperbolic group G, lifts to a quasihomomorphism $\tilde{f} : G \to H_{2n}$. We now explain how to use this in order to construct examples of quasihomomorphisms to nilpotent groups which are not close to homomorphisms.

It follows from the definition of H_{2n} that two elements $b, b' \in B$ commute if and only if $\omega(p(b), p(b')) = 0$. Take G which admits an epimorphism $f : G \to C' \cong \mathbb{Z}^2 < \mathbb{Z}^{2n}$ such that ω is nondegenerate on C' and $f^*(\omega)$ defines a trivial cohomology class of G. For instance, we can take G to be the fundamental group of a closed oriented surface of genus ≥ 2 and $f : G \to C$ induced by a map of nonzero degree $S \to T^2$. Or, in line with Thurston's suggestion mentioned in the introduction, we can take G to be the fundamental group of a closed hyperbolic 3-manifold M which admits a retraction $r : M \to S$ to a closed oriented hyperbolic surface $S \subset M$. (It follows from the work of Agol, Haglund and Wise that for every quasifuchsian surface subgroup of $\pi_1(S) < \pi_1(M)$ there exists a finite index subgroup of $\Gamma' < \pi_1(M)$ which retracts to $\pi_1(S) \cap \Gamma'$. Hence, examples which we need abound.) Then take the composition of r with a homomorphism induced by a nonzero degree map $S \to T$.

Lemma 2.12. Suppose that G is a hyperbolic group, we are given a central extension (8) and $f : G \to C$, a homomorphism such that $[f^*(\omega)] \neq 0$ in $H^2(G, \mathbb{Z})$. Then:

1. For each quasihomomorphism $\tilde{f} : G \to B$ as above, there is no finite index subgroup $G_o < G$ such that $\tilde{f}|G_o$ is within finite distance from a homomorphism.
2. The image of $\tilde{f}$ is not Hausdorff-close to an abelian subgroup of B.

Proof. 1. Suppose, for the sake of a contradiction, that there exists such $G_o < G$ and a homomorphism $f' : G_o \to B$ within finite distance from $f|G_o$. Then the distance between the homomorphisms $f_o := p \circ f'$ and $f|G_o$ is again bounded, which implies (since C is free abelian of finite rank) that the two homomorphisms are actually equal. Since G_o has finite index in G, the transfer argument shows that $[f_o^*(\omega)] = [f^*(\omega)] \in H^2(G_o, \mathbb{Z})$ is still nonzero. However, for arbitrary central extension

$$1 \to A \to \tilde{\Gamma} \to \Gamma \to 1$$

and arbitrary group Λ we have that a homomorphism $h : \Lambda \to \Gamma$ lifts to a homomorphism $\tilde{h} : \Lambda \to \tilde{\Gamma}$ if and only if the pull-back $h^*(\omega)$ of the extension cocycle, vanishes in $H^2(\Lambda, A)$. Thus, in our situation, we obtain a contradiction with the assumption about nontriviality of the $f^*(\omega)$.

2. Suppose that $\tilde{f}(G)$ is Hausdorff-close to an abelian subgroup $B' < B$. Then the subgroup $f(G) < C$ is Hausdorff-close to the abelian subgroup $C' = p(B')$. Since subgroups of the abelian group C are Hausdorff-close iff they are commensurable, we can assume, after replacing G with a finite index subgroup $G_o < G$, that $f(G_o)$ is contained in C' and, hence, $\tilde{f}(G_o)$ is contained in B'. As in Part 1, the restriction of the extension class ω to the finite index subgroup $C_o := f(G_o) < f(G)$ is still nontrivial. This, however, implies that each abelian subgroup of $p^{-1}(C_o)$, such as $B' \cap p^{-1}(C_o)$, projects to a cyclic subgroup of C, in particular, the restriction of ω to $p(B') = C_o$ is trivial in this case. A contradiction. □

Remark 2.13. As a warning to the reader, we note that, in general, even if B is finitely presented, its center may fail to be finitely generated, see e.g. [1].

Question 2.14. Is it true that for arbitrary (countable) abelian group A and a hyperbolic group G, every class in $H^2(G; A)$ is bounded, i.e., is represented by a cocycle taking only finitely many values?

Suppose that $\tilde{f}_1, \tilde{f}_2 : G \to B$ are distinct quasihomomorphisms lifting $f : G \to C$ and (8) is a central extension. Then for every $g \in G$

$$\tilde{f}_2(g) = \phi(g)\tilde{f}_1(g),$$

where $\phi(g) \in A$ (which we, as usual, identify with $i(A)$). It is immediate that $\phi : G \to A$ is a quasihomomorphism. We summarize these observation as

Lemma 2.15. Given a central extension (8), the following hold:

1. A homomorphism $f : G \to C$ lifts to a quasihomomorphism $\tilde{f} : G \to B$ if and only if the pull-back class $f^*([\omega]) \in H^2(G; A)$ is bounded.

2. Different quasihomomorphic lifts differ by quasihomomorphisms $G \to A$.

2.5. **Summary of constructions of quasihomomorphisms.** So far, we saw several basic constructions of quasihomomorphisms:

i) **Lift.** If $\bar{f} : G \to \bar{H}$ is a quasihomomorphism and $1 \to K \to H \to \bar{H} \to 1$ is a short exact sequence with a (virtually) abelian group K, then lift $\bar{f}$ (if possible) to a quasihomomorphism $f : G \to H$. Note that if the exact sequence quasisplits with a quasisplitting $s : \bar{H} \to H$, then we can always lift $\bar{f}$ to a quasihomomorphism $f = s \circ \bar{f}$. For instance, all almost homomorphisms $G \to H$ appear in this fashion.

ii) **Product.** If $f_i : G \to H_i$ are quasihomomorphisms, $i = 1, ..., n$, then take

$$f = (f_1, \ldots, f_n) : G \to H = \prod_{i=1}^{n} H_i.$$

iii) **Composition.** The special case of the composition construction is when $f : G \to H$ is a quasihomomorphism and $\iota : H \to \tilde{H}$ is a monomorphism; then we extend f to the quasihomomorphism $\tilde{f} = \iota \circ f$.

iv) **Extension from a finite index subgroup.** Extend $f_o : G_o \to H$ (if possible) to a quasihomomorphism $f : G \to H$, where $|G : G_o| < \infty$.

v) **Bounded perturbation.** Replace f (if possible) with a quasihomomorphism f' within finite distance from f. Note, however, that (unlike quasimorphisms to abelian groups) a bounded perturbation of a quasihomomorphism need not be a quasihomomorphism. For instance, we will show in Theorem 4.4 that if $f_1, f_2 : G \to H$ are quasihomomorphisms to a torsion-free hyperbolic group, and $\mathrm{dist}(f_1, f_2) < \infty$, then either $f_1 = f_2$, or f_1, f_2 are both bounded, or both are quasimorphisms to the same cyclic subgroup. Nevertheless, we will see and use repeatedly in the paper that sometimes quasihomomorphisms can be perturbed to quasihomomorphisms.

By using repeatedly these constructions one can obtain new quasihomomorphisms from a given set of quasihomomorphisms. In Theorem 1.2 we show that *all quasihomomorphisms are constructible*; in particular, there is no need to repeat the above constructions. Another construction which, as it turns out, to be not needed (in full generality) is the composition of quasihomomorphisms. One needs only its special cases as in (i) and (iii).

3. Rigidity of quasihomomorphisms

3.1. **Quasihomomorphisms and centralizers.** Consider a quasihomomorphism $f : G \to H$. By Part 1 of Lemma 2.5, there exists a finite subset $\{y_1, \dots, y_n\} \subset G' = f(G) \subset H$, such that for every $x \in G$ there exists y_j for which

$$\tilde{\psi}(x) = ad(y_j) \in Aut(\Delta_f),$$

i.e., for every $\delta \in \Delta_f$,

$$f(x)^{-1}\delta f(x) = y_j \delta y_j^{-1},$$

and, hence,

$$[f(x)y_j, \delta] = 1.$$

In other words, $f(x)y_j$ belongs to $Z_H(\Delta_f)$, the centralizer of Δ_f in H. Moreover, by Part 2 of the same lemma, if $\varphi = \varphi_f(x) = 1$ then we can choose $y_j \in \Delta_f$. Recall that the image of the homomorphism φ is finite and the kernel $G_o = \ker(\varphi)$ has finite index in G.

We, thus, obtain the following strengthening of Lemma 2.5:

Corollary 3.1. For every quasihomomorphism $f : G \to H$, there exists a constant C such that

$$f(G) \subset \mathcal{N}_C(Z_H(\Delta_f)).$$

Moreover, setting $G_o = \ker(\varphi)$, we get

$$f(G_o) \subset \bigcup_{i=1}^{n} Z_H(\Delta_f) \cdot y_i, \quad y_i \in \Delta_f.$$

In particular,

Corollary 3.2. Suppose that H has the property that the centralizer of every nontrivial element is abelian. Then for every quasihomomorphism $f : G \to H$ either f is a homomorphism or its image lies in a C-neighborhood of some abelian subgroup (with C depending on f, of course).

Example 3.3. Let H be either an (infinite) free Burnside group $B(n, m)$ on n generators and odd exponent $m \geq 665$, or a Tarski monster (see [26]), where all proper subgroups are finite cyclic. Note that by a theorem of Adyan and Novikov (see e.g. [26]), the centralizer of every nontrivial element of $B(n, m)$ is cyclic of order m. In the case of Tarski monsters constructed by Olshansky, centralizers of nontrivial elements are again cyclic, Theorem 26.5 of [26] (we owe the reference to Denis Osin). Therefore, for every group G, every unbounded quasihomomorphism $f : G \to H$ is a homomorphism. (Since if $D(f) \neq \{1\}$ then $f(G)$ is close to the centralizer of $D(f)$.)

Note, however, that for m even, some centralizers in $B(n, m)$ are infinite, see [20] for the details. This leads to

Question 3.4. Are there quasihomomorphisms $f : G \to H$ to torsion groups H, which are not within finite distance from almost homomorphisms?

We note that if H is a nilpotent torsion group, then indeed, the answer to this question is negative (since the defect subgroup is finite in this case). Furthermore, by repeating the construction in Example 2.11 with $A = \mathbb{Z}_2$ and G a countably infinite direct sum of $\mathbb{Z}_2$'s, it is easy to construct examples of quasihomomorphisms to torsion nilpotent groups which are not close to homomorphisms.

We next explain how one can alter f such that its image is actually contained in $Z_H(\Delta_f)$. As above, let $G_o = \ker(\varphi)$. We define a projection $r : f(G_o) \to Z_H(\Delta_f)$ by sending $h = f(g) = zy_i$ to z, where $z \in Z_H(\Delta_f)$,

$$y_i \in Y = \{y_1, \dots, y_n\} \subset \Delta_f.$$

Set

$$f_o := r \circ f : G_o \to Z_H(\Delta_f) < Z_H(\Delta_{f|G_o})$$

Clearly, $d(f, f_o) = R < \infty$, where $R = \max\{d(y, 1) : y \in Y\}$.

Lemma 3.5. The map f_o is a quasihomomorphism and $D(f_o) \subset \Delta_f$.

Proof. We have

$$f(x_1x_2) = f(x_1)f(x_2)s, \quad s \in D(f)$$

$$f(x_i) = f_o(x_i)\delta_i, \delta_i \in \Delta_f, f(x_1x_2) = f_o(x_1x_2)\delta_3, \quad |\delta_i| \le R, i = 1, 2, 3.$$

Since $f_o(x_i)$ commutes with Δ_f,

$$f_o(x_1)f_o(x_2)\delta_1\delta_2 = f_o(x_1)\delta_1 f_o(x_2)\delta_2 =$$

$$f(x_1)f(x_2) = f(x_1x_2)s = f_o(x_1x_2)\delta_3 s.$$

Therefore,

$$f_o(x_1)f_o(x_2) \sim_{D_o} f_o(x_1x_2)$$

where $D_o = D(f_o) \subset \Delta_f$ is finite (since $|\delta_i| \le R$ and $s \in D(f)$). ☐

We can now prove

Theorem 3.6. Every quasihomomorphism $f : G \to H$ is constructible: For the subgroup $G_o < G$ and the quasihomomorphism

$$f_o : G_o \to H_o < Z_H(\Delta_{f_o}) < H$$

as above, we have:

a) The projection of f_o to $\bar{f}_o : G_o \to Q = H_o/\Delta_{f_o}$ is a homomorphism.

b) $H_o = \langle f_o(G_o) \rangle$ and the finitely generated subgroup Δ_{f_o} is central in H_o.

Proof. Let $H_o < H$ be the subgroup generated by $f_o(G_o)$. By the construction,

$$f_o(G_o) \subset Z_H(\Delta_f) < Z_H(\Delta_{f_o})$$

since $\Delta_f > \Delta_{f_o}$. Since $H_o = \langle f_o(G_o)\rangle$, the subgroup $\Delta_{f_o} < H_o$ is central in H_o. Since Δ_{f_o} contains the defect set of f_o, the map $\bar{f}_o$ is a homomorphism. □

We note that Theorem 1.2 from the introduction follows immediately.

3.2. **Quasihomomorphisms close to abelian subgroups.** In this and the following section we establish two technical results, which are variations of Theorem 1.2 and will be used in the proof of Theorem 7.1.

Let B be a group which is an extension

$$1 \to A \to B \xrightarrow{p} C \to 1,$$

where A is a finitely generated abelian group. Suppose, further, that A is *virtually central* in B in the sense that there exists a finite index subgroup $C' \lhd C$ which acts trivially on A. We will then refer to B as a *virtually central extension of* C *by* A.

Proposition 3.7. Let B be a virtually central extension of C by A and $f : G \to B$ be a quasihomomorphism whose projection to C has bounded image. Then there exists a finite index subgroup $G_1 < G$ such that $f|G_1$ is within finite distance from a quasihomomorphism $f_1 : G \to A$ ($f_1 \sim f$). Furthermore, if A is contained in the center of B, then one can take $G_1 = G_o$, where $G_o < G$ is as in Theorem 3.6.

Proof. Let $\rho : C \to Aut(A)$ denote the action of C on A, let Q be the image of ρ; by our assumption, the group Q is finite. Without loss of generality, we may assume that the subset $f(G)$ generates B (otherwise, we replace B with $\langle f(G)\rangle$. By Theorem 3.6, there exists a finite-index subgroup $G_o < G$ and a quasihomomorphism $f_o : G_o \to B$ ($f_o \sim f|G_o$) such that Δ_{f_o} is contained in the center of B. In particular, $\rho p(\Delta_{f_o}) = \{1\}$ and, hence, the composition

$$G \xrightarrow{f_o} B \xrightarrow{p} C \xrightarrow{\rho} Q$$

is a homomorphism. Let G_1 denote the kernel of this homomorphism; it is a finite-index subgroup of G. By the construction, A is contained in the center of $B_1 = \ker(\rho \circ p)$. In what follows we use the restriction of the metric from B to B_1.

We let $r : B_1 \to A$ denote a nearest-point projection. We claim that the restriction of r_1 to each n-neighborhood $\mathcal{N}_n(A)$ of A in B_1 is a quasihomomorphism:

$$r(xy) \sim_{S_n} r(x)r(y)$$

for all $x, y, xy \in \mathcal{N}_n(A)$. The finite subsets S_n, in general, will depend on n.

The proof of the claim is similar to the one in the proof of Theorem 1.2. Let $h_i = a_i b_i \in B_1$, $a_i = r_o(h_i)$, $b_i \sim 1, b_i \in B_o, i = 1, 2$. Then, since A is central in B_1,

$$h_1 h_2 = a_1 a_2 b_1 b_2,$$

$$r(h_1 h_2) \sim a_1 a_2 = r(h_1) r(h_2),$$

cf. the proof of Lemma 3.5. Thus, the restriction of r to $\mathcal{N}_n(A)$ is indeed a quasihomomorphism. Consequently, the composition $f_1 = r \circ f_1 : G_1 \to A$ is also a quasihomomorphism. By the construction, the maps $f_1|G_1$ and $f|G_1$ are within finite distance from each other. Lastly, we note that if A is central in B, then $Q = 1$ and, thus, $B_1 = B, G_1 = G_o$. □

Corollary 3.8. Suppose that B is a finitely generated virtually abelian group, $B = A \rtimes C$, where A is free abelian of finite rank and C is finite. Then for each quasihomomorphism $f : G \to B$, there exists a finite-index subgroup $G_1 < G$ such that $f|G_1$ is within finite distance from a quasihomomorphism $f_1 : G_1 \to A$. Furthermore, if A is contained in the center of B, then one can take $G_1 = G_o$, where $G_o < G$ is as in Theorem 3.6.

3.3. **Quasihomomorphisms to finite extensions.** Suppose that we have an extension of a group Q, i.e., a short exact sequence

$$1 \to K \to H \xrightarrow{p} Q \to 1,$$

and a quasihomomorphism $f : G \to H$ such that $D(f)$ is contained in the center of H and $p \circ f(G)$ is finite, e.g., Q is a finite group. Assume, furthermore, that the subgroup $Q_o := p(\Delta_f)$ has finite index in Q.

Proposition 3.9. Under the above assumptions, there exists a finite index subgroup $G' < G$ and a quasihomomorphism $f' : G' \to K$, $f' \sim f$, $D(f') \subset \Delta_f$.

Proof. Since the subgroup Δ_f is central in H, its image $Q' = p(\Delta_f)$ is central in Q. The composition

$$G \xrightarrow{f} H \xrightarrow{p} Q \to Q/Q'$$

is then a homomorphism to a finite group; let G' denote its kernel. Since $p(\Delta_f) = Q'$ and $p \circ f(G)$ is finite, there exists a finite subset

$$D_1 = \{h_1, \ldots, h_n\} \subset \Delta_f,$$

such that

$$f(G') \subset \bigcup_{i=1}^{n} Kh_i.$$

Similarly to the proof of Proposition 3.7, we define the projection

$$r : \bigcup_{i=1}^{n} Kh_i \to K, \quad r(kh_i) = k.$$

Centrality of Δ_f in H implies that

$$k_1 h_{i_1} k_2 h_{i_2} = k_1 k_2 h_{i_1} h_{i_2} = k_1 k_2 h_{i_3},$$

with $h_{i_1}, h_{i_2} \in D_1$ and $h_{i_3} \in D_1^2$. It follows that $f' := r \circ f|G'$ is a quasihomomorphism and

$$D(f') \subset D_1^2 D(f) \subset \Delta_f.$$

Clearly, $\mathrm{dist}(f', f|G') < \infty$. □

4. Quasihomomorphisms to hyperbolic groups

Theorem 4.1. 1. Suppose that H is a torsion-free hyperbolic group. Then (for an arbitrary group G) every unbounded quasihomomorphism $f : G \to H$ is either a homomorphism or a quasimorphism to a cyclic subgroup of H.

2. Suppose that H is a general hyperbolic group. Then for every unbounded quasihomomorphism $f : G \to H$ either the image of f is contained in an elementary subgroup of H or f is an almost homomorphism.

Proof. In view of Corollary 3.1, $f(G)$ is contained in a C-neighborhood of the centralizer of Δ_f in H. Since $f(G)$ is infinite, it follows that the defect subgroup $\Delta = \Delta_f$ has infinite centralizer in H, and, hence, is elementary. By Lemma 2.3, $f(G)$ is contained in $N = N_H(\Delta)$, the normalizer of Δ in H. If Δ is finite then composition of f with the projection to $Q = N/\Delta$ is a homomorphism and, hence, f is an almost homomorphism. If Δ is infinite, then N is elementary. This concludes the proof of Part 2.

Suppose, furthermore, H is torsion free. If Δ is finite, then it is trivial and f is a homomorphism. If Δ is infinite, then N is infinite cyclic. Thus, $f : G \to N$ is a quasimorphism from G to an infinite cyclic subgroup of H. □

The following lemma is a sharpening of the statement about quasihomomorphisms to elementary groups:

Proposition 4.2. If $f : G \to H$ is an unbounded quasihomomorphism to an elementary hyperbolic group H, then, the reduction $\widehat{f}$ of f modulo the maximal finite normal subgroup $F \triangleleft H$ either is a quasimorphism (to $\mathbb{Z}$) or this statement holds after restricting $\widehat{f}$ to an index 2 subgroup $G_o < G$.

Proof. The projection of f, $\widehat{f} : G \to H/F$, is again a quasihomomorphism. Therefore, it suffices to consider the case when $F = 1$ and H is either $\mathbb{Z}$ or $\mathbb{Z}_2 \star \mathbb{Z}_2$; moreover, it suffices to consider the case where H is generated by $f(G)$. If $H \cong \mathbb{Z}$, then f is a quasimorphism. If $H \cong \mathbb{Z}_2 \star \mathbb{Z}_2$, the group Δ_f has to fix the ideal boundary of H pointwise (since it acts on H with bounded displacement). Therefore, the composition of f with the projection to $\mathbb{Z}_2$ is a homomorphism. Restricting f to the kernel G_o of this homomorphism results in a quasimorphism $\widehat{f} : G_o \to \mathbb{Z}$. □

Corollary 4.3. Suppose that Γ is an irreducible lattice in a semisimple Lie group of real rank ≥ 2. Then every quasihomomorphism $f : \Gamma \to H$, with hyperbolic target group H, is bounded.

Proof. First of all, it is proven in [7] (Corollary 1.3) that Γ has only bounded quasimorphisms. Suppose, therefore, that $f : \Gamma \to H$ is an unbounded quasihomomorphism. If the image of f is contained in an elementary subgroup of H then, after passing to an index 2 subgroup $\Gamma_o < \Gamma$, we obtain an unbounded quasimorphism $\Gamma_o \to \mathbb{Z}$ (see Proposition 4.2), which is a contradiction. Assume, therefore, that the subgroup $J' < H$ generated by $f(G)$ is nonelementary. According to Theorem 4.1, J' is contained in a subgroup $J < H$ which contains a finite normal subgroup $K \triangleleft J$ such that the projection of f to $\bar{J} = J/K$ is a homomorphism. Set $K' := K \cap J'$. Then the projection $\bar{f} : G \to \bar{J}' := J'/K'$ is a homomorphism as well. The subgroup $J' < H$ is nonelementary and the construction of quasimorphisms applied to $J' < H$ (see [15], [12]) yields unbounded quasimorphisms $h : J' \to \mathbb{Z}$. Since K' is a normal finite subgroup in J', the sequence

$$1 \to K' \to J' \to \bar{J}' \to 1$$

is quasisplit (see Example 2.8) and, hence, h projects to an unbounded quasimorphism $\bar{h} : \bar{J}' \to \mathbb{Z}$ (see the *projection* construction in §2.4.1). Composing the quasimorphism $\bar{h}$ with the homomorphism $\bar{f} : \Gamma \to \bar{J}'$, we obtain an unbounded quasimorphism $\Gamma \to \mathbb{Z}$, which again contradicts [7]. □

As another application of Theorem 4.1, we will prove *deformation rigidity* of quasihomomorphisms to torsion-free hyperbolic groups. It shows that a bounded perturbation such a quasihomomorphism is seldom a quasihomomorphism.

Theorem 4.4. Suppose that H is a torsion-free hyperbolic group and $f_1, f_2 : G \to H$ are quasihomomorphisms with $\mathrm{dist}(f_1, f_2) < \infty$. Then either both f_1, f_2 are bounded, or both take values in the same cyclic subgroup of H, or $f_1 = f_2$.

Proof. According to Theorem 4.1, each f_1, f_2 is either bounded, or is a quasimorphism to a cyclic subgroup or is a homomorphism. Recall that if C_1, C_2 are infinite cyclic subgroups of a hyperbolic group H then either their ideal boundaries in the Gromov boundary of H are disjoint, or C_1, C_2 generate an elementary subgroup of H. In the former case, for each $R < \infty$, the intersection

$$\mathcal{N}_R(C_1) \cap \mathcal{N}_R(C_2)$$

is bounded. In the setting of our theorem, it follows that if the image of f_i is contained in a cyclic subgroup C_i of H, then the image of f_{3-i} is contained in a cyclic subgroup of H containing C_i. Therefore, it remains to analyze the case when both f_1, f_2 are homomorphisms. For $x \in G$ let C_i denote the cyclic subgroup of H generated by $f_i(x)$. Since the homomorphisms

$$f_i : \langle x \rangle \to C_i < H$$

are within finite distance from each other, the subgroups C_1, C_2 generate a cyclic subgroup C of H. The reader will verify that if $f_i : \langle x \rangle \to C$ are two homomorphisms within finite distance from each other, they have to be equal. Hence, $f_1(x) = f_2(x)$ for all $x \in G$ when both f_1, f_2 are homomorphisms. □

5. Quasihomomorphisms to $CAT(0)$ groups

We will need several standard facts from the theory of $CAT(0)$ groups. We will use the notation $Isom(Y)$ for the isometry groups of metric spaces Y. From now on, we fix a $CAT(0)$ group Γ and a properly discontinuous cocompact isometric action $\Gamma \curvearrowright X$ of Γ on a $CAT(0)$ space X. (This action is not required to be faithful, but the kernel of the action is necessarily finite. We are unaware, though, of any examples of $CAT(0)$ groups which do not admit faithful properly discontinuous cocompact isometric actions on $CAT(0)$ spaces.) Recall that for an isometry α of X, the *displacement* of α is

$$D_\alpha = \inf_{y \in X} d(y, \alpha y).$$

Since $\Gamma \curvearrowright X$ is cocompact and properly discontinuous, for every $\alpha \in \Gamma$ this infimum is attained in X and one defines the *minimal set* Min_α of α as

$$\{x \in X : d(x, \alpha x) = D_\alpha\}.$$

It is clear that Min_α is closed; the $CAT(0)$ property implies that Min_α is convex [4, Ch II.6, Theorem 6.2] and, hence, is a $CAT(0)$ space. If α has infinite order, then Min_α splits isometrically as the product $Min_\alpha \cong \mathbb{R} \times X_1$, the isometry α acts trivially on X_1 and via a nontrivial translation on $\mathbb{R}$. Furthermore, Min_α equals the union of *axes* of α, i.e. α-invariant geodesics in X and each axis of α has the form $\mathbb{R} \times y$, $y \in X_1$. See [4, Ch II.6, Theorem 6.8]. Since for each α of infinite order, $\gamma \in \Gamma$ and an axis A of α, we have that γA is an axis of $\gamma\alpha\gamma^{-1}$, it follows that the normalizer of $\langle\alpha\rangle$ in Γ preserves Min_α and preserves its product decomposition $\mathbb{R}\times X_1$. Moreover, each element of the centralizer of α acts via a translation along the $\mathbb{R}$-factor of Min_α.

For an arbitrary subgroup $\Lambda < \Gamma$ we let Min_Λ denote the intersection

$$Min_\Lambda := \bigcap_{\alpha\in\Lambda} Min_\alpha.$$

This is a (a possibly empty) closed convex subset of X invariant under the normalizer $N_\Gamma(\Lambda)$ of Λ in Γ. The invariance property follows from

$$\gamma Min_\alpha = Min_{\gamma\alpha\gamma^{-1}}, \quad \alpha,\gamma\in\Gamma.$$

Lemma 5.1. Suppose that Γ is a $CAT(0)$ group and $\Gamma \curvearrowright X$ is a properly discontinuous isometric cocompact action on a $CAT(0)$ space X. Let $T < \Gamma$ be a finite subgroup. Then the fixed-point set

$$X^T = Min_T$$

of T in X is a nonempty closed convex subspace invariant under the normalizer $N_\Gamma(T)$ of T in Γ. Moreover, the quotient X^T/Γ^T is compact, where $\Gamma^T = Z_\Gamma(T) < N_\Gamma(T) < \Gamma$ is the centralizer of T in Γ. In particular, $N_\Gamma(T)$ is again a $CAT(0)$ group.

Proof. The fact that X_T is nonempty is a special case of the Cartan's Fixed Point Theorem (see [4, Ch. II.2, Corollary 2.8]). Compactness of X_T/Γ_{X_T} is proven in [30, Remark 2]. □

Recall that abelian subgroups of $CAT(0)$ groups are finitely generated, see [4, Ch II.7, Corollary 7.6].

Lemma 5.2. Suppose that X_1 is a $CAT(0)$ space $\Gamma_1 \curvearrowright X_1$ is a properly discontinuous isometric action, $A_1 < \Gamma_1$ is a free abelian subgroup of Γ_1. Then:

1. Min_{A_1} is nonempty, invariant under the normalizer $N_{\Gamma_1}(A_1)$ and the action of $Z_{\Gamma_1}(A_1)$ is cocompact on Min_{A_1}. In particular, the normalizer $N_{\Gamma_1}(A_1)$ is a $CAT(0)$ group.

2. Furthermore, the minimal set Min_{A_1} splits isometrically as $E\times Y$ where E is a finite-dimensional Euclidean space, the splitting is invariant under $N_{\Gamma_1}(A_1)$. The group $N_{\Gamma_1}(A_1)$ acts cocompactly on Y with kernel containing A_1 and the action of $N_{\Gamma_1}(A_1)/A_1$ on Y is properly discontinuous.

Proof. We note that the existence of the $N_{\Gamma_1}(A_1)$-invariant decomposition $Min_{A_1}\cong E\times Y$ is proven in [4, Ch II.7, Theorem 7.1]. The same theorem shows that for each $y\in Y$ the group A_1 acts cocompactly on $E\times\{y\}$. The quotient space $Q = Min_{A_1}/A_1$ fibers over Y_1 with compact fibers and the group $N_{\Gamma_1}(A_1)/A_1$ acts on Q properly discontinuously. This implies proper discontinuity of the action of $N_{\Gamma_1}(A_1)/A_1$ on Y_1. Once we know that $Z_{\Gamma_1}(A_1)$ acts cocompactly on Min_{A_1}, cocompactness of the action of $Z_{\Gamma_1}(A_1)/A_1$ on Y_1 will follow.

Remark 5.3. Note that the kernel of the action of $N_{\Gamma_1}(A_1)$ on Y_1 could be larger than A_1 because of the kernel of the action of $N_{\Gamma_1}(A_1)$ on Min_{A_1}.

Thus, we only have to prove Part 1 of the lemma. The proof is by induction on the rank of A_1 (which is necessarily finite). Suppose first that $A_1\cong\mathbb{Z}$. The cocompactness of the action $Z_{\Gamma_1}(A_1) \curvearrowright Min_{A_1}$ in this case is proven in [30, Theorem 3.2]. We assume that the claim holds for all $CAT(0)$ spaces X, properly discontinuous cocompact isometric actions $\Gamma \curvearrowright X$ and free abelian subgroups $A < \Gamma$ of rank $n-1$. Suppose that the group A_1 in the lemma has rank n. We split A_1 as the product $A_2' \times A_2$, where $A_2' \cong \mathbb{Z}$, $\mathrm{rank}(A_2) = n-1$. Then, by applying [30, Theorem 3.2] to the subgroup $A_2' < \Gamma_1$, we obtain that the group $Z_{\Gamma_1}(A_2')$ (containing $Z_\Gamma(A_1)$) acts cocompactly on $Min_{A_2'}$. As we noted earlier, the group

$$\widetilde{\Gamma}_2 := N_{\Gamma_1}(A_2')$$

preserves the subset $Min_{A_2'} \subset X_1$ and its product decomposition $\mathbb{R} \times X_2$. We consider the restriction homomorphism

$$\rho : \widetilde{\Gamma}_2 \to Isom(Min_{A_2'}).$$

The kernel of this homomorphism is finite and, hence, the centralizer $Z_{\Gamma_1}(A_2) < \widetilde{\Gamma}_2$ maps to a finite index subgroup in the centralizer of $\rho(A_2)$ in $\rho(\widetilde{\Gamma}_2)$:

$$|Z_{\rho(\widetilde{\Gamma}_2}(\rho(A_2) : \rho(Z_{\Gamma_1}(A_2))| < \infty. \tag{9}$$

Since $Min_{A_2'}$ is closed and convex in X_1, and the nearest-point projection $X_1 \to Min_{A_2'}$ is distance nonincreasing, it follows that for each $\gamma \in \widetilde{\Gamma}_2$ we have

$$Min_\gamma \subset Min_{A_2'}. \tag{10}$$

The action of $\widetilde{\Gamma}_2$ on the X_2-factor of $Min_{A_2'}$ defines a homomorphism $\phi : \widetilde{\Gamma}_2 \to Isom(X_2)$ whose image we will denote by Γ_2. Since the actions of A_2' on $\mathbb{R}$ and of $\widetilde{\Gamma}_2$ on $Min_{A_2'}$ are cocompact, the action $\Gamma_2 \curvearrowright X_2$ is properly discontinuous and cocompact as well.

We now apply the induction hypothesis to the action $\Gamma_2 \curvearrowright X_2$ and the abelian subgroup $A_2'' := \phi(A_2) < \Gamma_2$. The subset $Min_{A_2''} \subset X_2$ is nonempty and the action of $Z_{\Gamma_2}(A_2'')$ is cocompact on Min_{A_2}. The preimage of $Min_{A_2''}$ in $Min_{A_2'}$ under the projection

$$p : Min_{A_2'} \cong \mathbb{R} \times X_2 \to X_2$$

is contained in the minimal set Min_{A_1}, see (10). Since A_1 centralizes A_2', it acts via translations on the $\mathbb{R}$-factor of $\mathbb{R} \times X_2$. Therefore,

$$p^{-1}(Min_{A_2''}) = Min_{A_1}.$$

Since, by the induction assumption, $Min_{A_2''}/Z_{\Gamma_2}(A_2'')$ is compact, taking into account (9), we conclude that the group $Z_{\Gamma_1}(A_1)$ acts cocompactly on Min_{A_1}. Lemma follows. □

We now can describe the structure of normalizers and centralizers of abelian subgroups of $CAT(0)$ groups.

Proposition 5.4. Suppose that $\Gamma \curvearrowright X$ is a cocompact properly discontinuous action of Γ on a $CAT(0)$ space X and let $A < \Gamma$ be a finitely generated abelian subgroup with the torsion subgroup $T < A$. Then the centralizer $\Gamma' := Z_\Gamma(A)$ and the normalizer $N_\Gamma(A)$ of A in Γ satisfy the following:

1. $N_\Gamma(A)$ preserves a closed convex nonempty subset $C \subset X$ such that $Z_\Gamma(A)$ acts on C properly discontinuously and cocompactly. In particular, both $Z_\Gamma(A)$ and $N_\Gamma(A)$ are $CAT(0)$ groups and $Z_\Gamma(A)$ has finite index in $N_\Gamma(A)$.

2. The short exact sequence

$$1 \to A \to \Gamma' \to \Gamma'/A \to 1$$

virtually splits in the following sense: The quotient $\Gamma_1 = \Gamma'/\Phi$ of Γ' by a finite normal subgroup Φ containing T, contains a finite index subgroup Γ'_o isomorphic to $A_1 \times \Pi_o$, where $A_1 \cong A/T$.

3. Furthermore, Π_o is also $CAT(0)$ group and there exists a properly discontinuous cocompact isometric action $\Pi_o \curvearrowright Y$ on a nonempty $CAT(0)$ space Y, such that Y is isometric to a closed convex subset of X.

Proof. The torsion subgroup $T < A$ is invariant under the action of $N_\Gamma(A)$ by conjugation and, hence, $N_\Gamma(A) < N_\Gamma(T)$. Applying Lemma 5.1 we obtain a closed convex nonempty subset $X^T \subset X$ invariant under $N_\Gamma(T)$, on which $N_\Gamma(T)$ acts cocompactly. Consider the restriction homomorphism

$$\rho : N_\Gamma(T) \to Isom(X^T),$$

whose kernel Φ (containing T) is necessarily finite. This homomorphism defines a properly discontinuous cocompact action of $\Gamma_1 := \rho(N_\Gamma(T))$ on $X_1 := X^T$. In particular, Γ_1 is a $CAT(0)$ group, $A/T \cong A_1 := \rho(A) < \Gamma_1$ is a free abelian group of finite rank. The centralizer and the normalizer of A in Γ map via ρ respectively into the centralizer and the normalizer of A_1 in Γ_1. Furthermore,

$$|N_{\Gamma_1}(A_1) : \rho(N_\Gamma(A))| < \infty, \quad |Z_{\Gamma_1}(A_1) : \rho(Z_\Gamma(A))| < \infty. \tag{11}$$

We now consider the free abelian subgroup $A_1 < \Gamma_1$ of the $CAT(0)$ group Γ_1. In view of Lemma 5.2, the groups $N_{\Gamma_1}(A_1)$ and $Z_{\Gamma_1}(A_1)$ act properly discontinuously and cocompactly on the closed convex subset $C := Min_{A_1} \subset X_1 \subset X$. This subset is invariant under $N_\Gamma(A)$ and taking into account (11), the first claim of the proposition follows.

Since the group A_1 is free abelian of finite rank, [4, Ch II.7, Theorem 7.1] implies the existence of a finite index subgroup $\Gamma'_o < \Gamma_1$ isomorphic to $\Pi_o \times A_1$. This proves the second claim.

To prove the last claim of the proposition, we apply Part 2 of Lemma 5.2: The $CAT(0)$ space $Min_{A_1} \subset X_1$ splits isometrically as $E \times Y$ and the isometric action of Γ_1/A_1 on Y is properly discontinuous and cocompact. Since Π_o maps to a finite index subgroup of Γ_1/A_1, the group Π_o acts acts on Y properly discontinuously and cocompactly. Lastly, Y embeds isometrically as a cross-section $e \times Y$ ($e \in E$) of the closed convex subset $Min_{A_1} \cong E \times Y$ of X. □

We can now prove a rigidity theorem for quasihomomorphisms to $CAT(0)$ groups:

Theorem 5.5. Suppose that H is a $CAT(0)$ group. Then for every quasihomomorphism $f : G \to H$ there exists a finite-index subgroup $G^o < G$, a $CAT(0)$ subgroup $H' < H$, a finite normal subgroup $\Phi < H'$ and a quasihomomorphism $f^o : G^o \to H' < H$ within finite distance from $f|G^o$ such that the projection $\bar{f}^o$ of f^o to H'/Φ splits as a product map

$$f^o = (f_1, f_2) : G^o \to H_1 \times H_2 < H'/F,$$

where $f_1 : G^o \to H_1$ is a homomorphism to a $CAT(0)$-group and f_2 is a quasihomomorphism to a finitely generated free abelian group H_2.

Proof. We continue with the notation in Theorem 3.6. We obtain a finite index subgroup $G_o < G$ and a quasihomomorphism

$$f_o : G_o \to H_o := Z_H(\Delta_{f_o}) < H$$

within finite distance from $f|G_o$. We let A denote the (finitely generated) abelian group Δ_{f_o} and $T < A$ the torsion subgroup. We have quotient homomorphisms

$$H_o \xrightarrow{p} H_o/T \xrightarrow{q} H_o/A.$$

By Proposition 5.4, there exists a finite normal subgroup Φ of H_o containing T such that the quotient group H_o/Φ contains a finite index subgroup H^o which splits as the product $H_1 \times H_2$, where $H_1 = \Pi_o$ is a $CAT(0)$ group and $H_2 \cong A/T$. Since A contains the defect set of f_o, the composition $h := q \circ p \circ f_o$ is a homomorphism.

Setting $H' := p^{-1}(H^o) < H_o$, we conclude that $G^o := h^{-1}(q(H_o)) < G_o$ is a finite index subgroup of G. Then we obtain a quasihomomorphism

$$f^o := p \circ f_o = (f_1, f_2) : G^o \to H_1 \times H_2,$$

where f_1 is a homomorphism and $f_2 : G^o \to H_2$ is a quasihomomorphism to a free abelian group. □

Corollary 5.6. Suppose that H is a uniform lattice in a connected reductive algebraic Lie group and G is an irreducible lattice in a semisimple algebraic Lie group of real rank ≥ 2. Then for every quasihomomorphism $f : G \to H$ there exists a finite index subgroup $G^o < G$ and a quasihomomorphism $\tilde{f} : G^o \to H$ within finite distance from $f|G^o$ such that $\tilde{f}$ is an almost homomorphism.

Proof. The group H is a $CAT(0)$ group, acting (with finite kernel) on a certain nonpositively curved symmetric space. We thus can apply Theorem 5.5 (whose notation we will be now using). The subgroup $G^o < G$ is still an irreducible higher rank lattice; therefore, it has only bounded quasihomomorphisms to free abelian groups (see [7]). Hence, the map f_2 in Theorem 5.5 is bounded and

$$\mathrm{dist}(f^o, f_1) < \infty,$$

$$f_1 : G^o \to H_1 < H'/\Phi$$

is a homomorphism, where $\Phi < H'$ is a finite normal subgroup. Since Φ is finite, the map f_1 lifts to an almost homomorphism $\tilde{f} : G^o \to H' < H$. By the construction, the maps $f|G^o, \tilde{f}$ are finite distance apart. □

Example 5.7. There are higher rank (non-residually finite) uniform lattices H as in Corollary 5.6 with finite nontrivial center $Z_H < H$, such that Z_H is contained in every finite index subgroup of H, see [29]. (The group H is a lattice in a nonlinear connected algebraic Lie group, a $\mathbb{Z}_2$-central extension of the group $SO(n, 2)$.) Therefore, setting $G = H/Z_H$ and letting $f : G \to H$ be a (quasihomomorphic) lift of the identity homomorphism $G \to H/Z_H$, we obtain examples of quasihomomorphisms whose restrictions to any finite index subgroup $G_o < G$ are not close to homomorphisms $G_o \to H$.

Theorem 5.8. Suppose that G is a connected semisimple algebraic Lie group of rank ≥ 2 without nontrivial compact normal subgroups and $\Gamma < G$ is an irreducible lattice. Then each quasihomomorphism $f : \Gamma \to \Gamma$ either has bounded image or is an automorphism of Γ.

Proof. In view of Theorem 1.2, after replacing Γ with a finite index subgroup Γ_o and $f|\Gamma_o$ with a nearby quasihomomorphism f_o, we obtain:

$$f_o : \Gamma_o \to \Lambda < \Gamma, \quad 1 \to A \to \Lambda \xrightarrow{p} Q \to 1,$$

where A is a central subgroup of a subgroup $\Lambda < \Gamma$, containing Δ_{f_o} and $f' := p \circ f_o$ is a homomorphism. We let $\bar{\Lambda}$ denote the Zariski closure of Λ in G; we will use the notation $\bar{A}$ for the Zariski closure of A in G.

By the Margulis Normal Subgroups Theorem, each nontrivial normal subgroup of Γ_o has finite index in Γ_o. (Here we are using the fact that Γ_o is Zariski dense in G and G has no nontrivial compact normal subgroups.) We apply this to the kernel $\ker(f')$ of f'.

1. If $\Gamma^o := \ker(f')$ has finite index in Γ_o, the restriction of f_o to this kernel is a quasihomomorphism $\Gamma^o \to A$. According to [7], $f_o|\Gamma^o$ is bounded; hence, f is bounded as well.

2. Assume that f' is a monomorphism. Then f_o projects to a monomorphism of Γ_o to the algebraic group $G_1 = \bar{\Lambda}/\bar{A}$. By the Margulis Superrigidity Theorem, the restriction of f_o to a finite index subgroup of Γ is induced by an injective homomorphism $G \to G_1$. The group A has to be finite (since $\dim(G_1) \geq \dim(G)$). If $\bar{A}$ is nontrivial, then the dimension of $\bar{\Lambda}$ (and, hence, of G_1) is strictly smaller than the one of G (since G has no nontrivial normal compact subgroups). We conclude that $A = \{1\}$ and, hence, $f_o : \Gamma_o \to \Gamma < G$ is a monomomorphism whose image necessarily has finite index in Γ (say, by the Mostow Rigidity Theorem). Thus, the image $f(\Gamma) < \Gamma$ is Hausdorff-close to the subgroup Γ. By Corollary 3.1, $f(\Gamma)$ is contained in a C-neighborhood of the centralizer of Δ_f in Γ. Since the centralizer of a nontrivial element of Γ has infinite index in Γ, it follows that $\Delta_f = \{1\}$, i.e., f is a homomorphism, which is necessarily injective. By the Mostow Rigidity Theorem f is induced by an automorphism of G and, hence, $f(\Gamma) = \Gamma$. □

6. Mapping class groups

In this section we collect some definitions and facts about mapping class groups of surfaces of finite type that will be used in the following section in order to prove a rigidity theorem for quasihomomorphisms to mapping class groups. Most of this material is quite standard, we refer the reader to [13, 21] for the details.

6.1. **Basic definitions.** A *finite type* surface Σ is an oriented (possibly disconnected) surface (without boundary), admitting a complete hyperbolic metric of finite area. A*peripheral loop* in Σ is a simple loop $\alpha \subset \Sigma$ such that one of the components of $\Sigma \setminus \alpha$ is an annulus. An simple loop $c \subset \Sigma$ is *essential* if it is not peripheral and does not bound a disk in Σ. More generally, an *essential multiloop* on Σ is a disjoint union of pairwise nonisotopic essential loops in Σ. A subsurface $\Sigma' \subset \Sigma$ is called *essential* if each essential loop in Σ' is still essential in Σ.

We let $Map(\Sigma)$ denote the *mapping class group* of Σ,

$$Map(\Sigma) = Homeo(\Sigma)/Homeo_o(\Sigma),$$

where $Homeo_o(\Sigma)$ is the connected component of the identity map $\Sigma \to \Sigma$ in the full group of homeomorphisms $Homeo(\Sigma)$. For $a \in Map(\Sigma)$ we let $h_a \in Homeo(\Sigma)$ denote an (unspecified) homeomorphism representing a.

We let $PMap(\Sigma) < Map(\Sigma)$ denote a finite index normal subgroup equal to the kernel of the homomorphism

$$Map(\Sigma) \longrightarrow Aut(H_1(\Sigma, \mathbb{Z}/3)),$$

defined via the action of homeomorphisms of Σ on its 1st homology group. We will refer to $PMap(\Sigma)$ as the *pure subgroup* of $Map(\Sigma)$. The pure subgroup entirely consists of *pure mapping classes*. We will discuss pure mapping classes in more detail in §6.2. For now we only note that each $a \in PMap(\Sigma)$ obviously acts trivially on $H_0(\Sigma)$ and preserves isotopy classes of all peripheral loops and that the subgroup $PMap(\Sigma)$ is torsion-free.

Given an essential multiloop $c \subset \Sigma$, define the *twist subgroup* $T_c < PMap(\Sigma)$ associated to c, to be the group generated by the Dehn twists along the components of c. Then T_c is a free abelian group of rank r, where r is the number of components of c.

For an essential multiloop $c \subset \Sigma$ we let $Map_c(\Sigma) < Map(\Sigma)$ denote the subgroup consisting of mapping classes which preserve c (but are allowed to permute components of c and to change the orientation of some of the components). The twist subgroup T_c is a normal subgroup in $Map_c(\Sigma)$.

If

$$\Sigma = \Sigma_1 \sqcup ... \sqcup \Sigma_m$$

is a decomposition of Σ into its connected components, then the group $Map(\Sigma)$ contains the product

$$\prod_{i=1}^{m} Map(\Sigma_i)$$

as a finite index normal subgroup with the quotient group $Q < S_n$ (the group Q acts on Σ by permuting homeomorphic components of Σ). In the context of pure subgroups, we have

$$PMap(\Sigma) \cong \prod_{i=1}^{m} PMap(\Sigma_i).$$

6.2. **Reduction systems and pure elements of $Map(\Sigma)$.** According to the Nielsen–Thurston classification, for a connected surface Σ all elements of $Map(\Sigma)$ are classified as:

1. Finite order.
2. Reducible.
3. Pseudo-Anosov.

Each torsion subgroup of $Map(\Sigma)$ is finite, since the pure subgroup $PMap(\Sigma)$ is torsion-free.

Lemma 6.1. Suppose that Σ is connected. Then the normalizer $N_{Map(\Sigma)}(\langle a\rangle)$ for each pseudo-Anosov element $a \in Map(\Sigma)$ is virtually infinite cyclic,

$$|N_{Map(\Sigma)}(\langle a\rangle) : \langle a\rangle| < \infty.$$

The centralizer $Z_{PMap(\Sigma)}(\langle a\rangle)$ of a in the pure mapping class group is infinite cyclic, consisting only of pseudo-Anosov elements (and the identity).

Proof. A proof can be found for instance in [24]. Alternatively, the statement about centralizers in $PMap(\Sigma)$ is the content of [21, Lemma 8.13]; the statement about the normalizer follows by taking the intersection

$$Z_{PMap(\Sigma)}(\langle a\rangle) = N_{Map(\Sigma)}(\langle a\rangle) \cap PMap(\Sigma),$$

which has finite index in $N_{Map(\Sigma)}(\langle a\rangle)$. □

Remark 6.2. One also has $N_{PMap(\Sigma)}(\langle a\rangle) \cong \mathbb{Z}$, but we will not need this property.

Corollary 6.3. Suppose that Σ has the connected components $\Sigma_1, \dots, \Sigma_m$, $a_i \in Map(\Sigma_i)$ are pseudo-Anosov, $i = 1, \dots, m$; define the free abelian subgroup $A < Map(\Sigma)$ generated by $a_1, \dots, a_m$. Then

$$Z_{PMap(\Sigma)}(A) \cong \mathbb{Z}^m.$$

Each reducible element $a \in Map(\Sigma)$ admits a *canonical reduction system* (see e.g. [21, §7.4]), which is a certain essential multiloop $c_a \subset \Sigma$ invariant under h_a (the orientation on some of the loops can be reversed). Due to the canonical nature of c_a, this multiloop is invariant (up to isotopy) under the normalizer $N_{Map(\Sigma)}(\langle a\rangle)$. The multiloop c_a has the property that (up to isotopy) it is contained in each h_a-invariant multiloop in Σ.

An element $a \in Map(\Sigma)$ is *pure* if it is orientation-preserving and either it is pseudo-Anosov or it is reducible, so that h_a preserves (up to isotopy) each component of c_a (together with its orientation), preserves all complementary components $\Sigma_i \subset \Sigma \setminus c_a$, and the restriction of h_a to each Σ_i defines either the trivial or a pseudo-Anosov element of $Map(\Sigma_i)$. A pure reducible element of $Map(\Sigma)$ is trivial iff c_a is empty. Minimality of c_a implies that if $a \in Map(\Sigma)$ is pure and preserves (up to isotopy) an essential subsurface $\Sigma' \subset \Sigma$, then a preserves each component and each boundary loop of Σ. The subgroup $PMap(\Sigma)$ consists only of pure elements, see [21, Corollary 1.8].

6.3. **Mapping class groups of surfaces with boundary.** Suppose that $\widehat{\Sigma}$ is a surface with nonempty boundary C, which is a partial compactification of a finite type surface $\Sigma = \widehat{\Sigma} \setminus C$. In this setting one defines the *relative mapping class group* $Map(\widehat{\Sigma}, C)$ as the quotient,

$$Homeo(\widehat{\Sigma}, C)/Homeo_o(\widehat{\Sigma}, C),$$

where $Homeo(\widehat{\Sigma}, C)$ is the group of homeomorphisms of Σ fixing the boundary C pointwise, and $Homeo_o(\widehat{\Sigma}, C) < Homeo(\widehat{\Sigma}, C)$ is the identity component. We define the pure mapping class group $PMap(\widehat{\Sigma}, C)$ analogously to the case of mapping class groups for surfaces without boundary, as the kernel of the homomorphism

$$Map(\widehat{\Sigma}, C) \to Aut(H_1(\Sigma, \mathbb{Z}/3)).$$

The inclusion $\Sigma \hookrightarrow \widehat{\Sigma}$ defines the restriction homomorphism

$$Homeo(\widehat{\Sigma}) \to Homeo(\Sigma)$$

and the associated homomorphism of mapping class groups

$$\rho : Map(\widehat{\Sigma}, C) \to Map(\Sigma).$$

The homomorphism ρ is neither surjective nor injective: Its image is a finite index normal subgroup of $Map(\Sigma)$ which is contained in the subgroup $Map_+(\Sigma)$ consisting of orientation preserving mapping classes. The quotient $Map_+(\Sigma)/\rho(Map(\widehat{\Sigma}, C))$ is isomorphic to the permutation group S_n, where n is the number of the components of C. Indeed, every orientation-preserving homeomorphism of Σ preserving each end of Σ is isotopic to a homeomorphism which extends to an element of $Homeo(\widehat{\Sigma}, C)$. Conversely, each permutation of components of C is realizable by an orientation-preserving homeomorphism $\widehat{\Sigma} \to \widehat{\Sigma}$.

The kernel of ρ is a free abelian subgroup T_C of rank n, its free basis consists of Dehn twists D_{α_i} along loops $\alpha_i \subset \Sigma$, parallel to the components of C, $i = 1, \dots, n$. By restricting to the pure mapping class groups we obtain a short exact sequence

$$1 \to T_C \to PMap(\widehat{\Sigma}, C) \to PMap(\Sigma) \to 1. \tag{12}$$

Proposition 6.4. The sequence (12) quasisplits.

Proof. The proof is by induction on the number n of components of C.

1. Suppose that $n = 1$, i.e., C is connected. Let S denote the surface closed surface obtained from $\widehat{\Sigma}$ by attaching the 2-disk along C. In this case, the obstruction to splitting the sequence (12) is the Euler class $e \in H^2(PMap(\Sigma); \mathbb{Z})$, which can be defined as the pull-back of the Euler class

$$\tilde{e} \in H^2(Homeo(S^1); \mathbb{Z})$$

under the embedding

$$PMap(\Sigma) \to Aut(\pi_1(S)) \to Homeo(S^1),$$

see [13, Section 5.5.4]. The class $\tilde{e}$ is bounded, see e.g. [17]. Therefore, the class e is bounded as well. Hence, the sequence (12) quasisplits.

2. Suppose that the claim holds for all surfaces with $n-1$ boundary components. Let $\widehat{\Sigma}$ be a surface with

$$\partial\widehat{\Sigma} = C = C_1 \sqcup \ldots \sqcup C_n.$$

Define the surface $\widehat{\Sigma}'$ by removing the circle C_n from $\widehat{\Sigma}$ and set $C' := C \setminus C_n = \partial\widehat{\Sigma}'$. The surface $\widehat{\Sigma}'$ has $n-1$ boundary components, hence, by the induction hypothesis, there exists a quasisplitting

$$s' : PMap(\Sigma') \to PMap(\widehat{\Sigma}', C'),$$

of the central extension

$$1 \to T_{C'} \to PMap(\widehat{\Sigma}', C') \to PMap(\Sigma) \to 1.$$

We claim that the central extension

$$1 \to T_{C_n} \to PMap(\widehat{\Sigma}, C) \to PMap(\widehat{\Sigma}', C') \to 1 \tag{13}$$

quasisplits, equivalently, has bounded extension class. Given a quasisplitting

$$s'' : PMap(\widehat{\Sigma}', C') \to PMap(\widehat{\Sigma}, C),$$

we then compose it with a quasisplitting s' as above and obtain a quasisplitting

$$s = s'' \circ s' : PMap(\Sigma') \to PMap(\widehat{\Sigma}, C)$$

of (12).

To prove existence of s'' we use the following trick. Define a new surface S by attaching one-holed tori $R_1, \ldots, R_{n-1}$ to $\widehat{\Sigma}$ along each circle $C_1, \ldots, C_{n-1}$ (leaving the last circle C_n untouched). The surface S now has only one boundary circle. Each homeomorphism

$$h \in Homeo(\widehat{\Sigma}, C)$$

extends to a homeomorphism $\tilde{h}$ of S by the identity on each R_i. Projecting $\tilde{h}$ to the mapping class group $Map(S, \partial S)$, yields embeddings

$$j : Map(\widehat{\Sigma}, C) \hookrightarrow Map(S, C_n)$$

and the analogous embedding

$$j : Map(\widehat{\Sigma}', C') \hookrightarrow Map(S')$$

for the surface $S' := S \setminus C_n$ (which has empty boundary). We obtain a commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & T_{C_n} & \longrightarrow & PMap(\widehat{\Sigma}, C) & \longrightarrow & PMap(\widehat{\Sigma}', C') & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j'} & & \\
1 & \longrightarrow & T_{C_n} & \longrightarrow & PMap(S, C_n) & \longrightarrow & PMap(S') & \longrightarrow & 1
\end{array}$$

We now apply the 1st step of induction to the bottom row of this diagram to obtain a quasisplitting σ of that central extension. Restricting σ to $PMap(\widehat{\Sigma}', C')$ we obtain the desired quasisplitting of the top row of the diagram, i.e., of the central extension (13). ☐

6.4. **Reducible subgroups.** Recall that for each essential multiloop $c \subset \Sigma$, we have two subgroups of $Map(\Sigma)$: The subgroup $Map_c(\Sigma)$ and its normal subgroup T_c (the twist subgroup). The subgroup $PMap_c(\Sigma) := Map_c(\Sigma) \cap PMap(\Sigma)$ still contains T_c. Define the essential subsurface $\Sigma_c := \Sigma \setminus c$.

Lemma 6.5. The inclusion $T_c \hookrightarrow PMap_c(\Sigma)$ defines a short exact sequence

$$1 \to T_c \to PMap_c(\Sigma) \xrightarrow{\pi} PMap(\Sigma_c) \to 1.$$

Proof. The homomorphism $\pi : PMap_c(\Sigma) \to PMap(\Sigma_c)$ is induced by restricting homeomorphisms of Σ preserving c to the subsurface Σ_c. The fact that its kernel contains T_c is immediate. We next prove the equality. Let $\mathcal{N}(c) \subset \Sigma$ denote an open regular neighborhood of c in Σ; the inclusion

$$\Sigma \setminus \mathcal{N}(c) \hookrightarrow \Sigma_c$$

is a homotopy-equivalence. If $f \in Homeo(\Sigma)$ fixes $\Sigma \setminus \mathcal{N}(c)$ pointwise, then f projects to an element of the twist subgroup T_c. It follows that $\ker(\pi) = T_c$.

To prove surjectivity of π, we note that each element of

$$a \in PMap(\Sigma_c) \cong PMap(\Sigma \setminus \mathcal{N}(c))$$

can be represented by a homeomorphism h_a of $\Sigma \setminus \mathcal{N}(c)$ fixing the boundary of this subsurface pointwise. We then extend h_a to each annular component of $\mathcal{N}(c)$ by an iterated Dehn twist. The result is a homeomorphism $\tilde{h}_a$ of Σ preserving c, and projecting to an element $\tilde{a} \in PMap_c(\Sigma)$ such that $\pi(\tilde{a}) = a$. ☐

6.5. **Structure of infinite abelian subgroups and their normalizers.** The structure of infinite abelian subgroups $A < Map(\Sigma)$ is described in [3] and in [21, chapter 8]. Below is a brief review of this description, where we limit ourselves to the setting of pure subgroups of mapping class groups. The intersection $A_P := A \cap PMap(\Sigma)$ is a finite index subgroup of A; this subgroup is either cyclic pseudo-Anosov, or A_P contains nontrivial reducible elements. We consider the latter case. For any nontrivial reducible elements $a_1, a_2 \in A_P$, the multiloops c_{a_1}, c_{a_2} are disjoint up to an isotopy, but some of the components of these multiloops could be isotopic to each other. We pick an auxiliary complete hyperbolic metric on Σ and let c_A denote the union of closed geodesics in Σ representing all the loops in c_a, where $a \in A_P$ are nontrivial reducible elements. In order to simplify the notation, in what follows we will denote c_A by c. Then c is an essential multiloop in Σ invariant under A_P. Due to the canonical nature of c, this multiloop is invariant (up to isotopy) under the normalizer of A in $Map(\Sigma)$. Restricting to $PMap(\Sigma)$, we conclude that the normalizer $N_{PMap(\Sigma)}(A_P)$ of A_P in $PMap(\Sigma)$ is a subgroup of $Map_c(\Sigma)$. Since all the elements of $PMap(\Sigma)$ are pure, they have to preserve each component of c and its orientation. We obtain

$$T_c < Z_{PMap(\Sigma)}(A_P) < N_{PMap(\Sigma)}(A_P) < PMap_c(\Sigma).$$

By restricting the homomorphism π defined in the previous section to the subgroup $N_{PMap(\Sigma)}(A_P)$, we obtain the homomorphism

$$N_{PMap(\Sigma)}(A_P) \xrightarrow{\pi} PMap(\Sigma_c)$$

and the exact sequence

$$1 \to T_c \to N_{PMap(\Sigma)}(A_P) \xrightarrow{\pi} PMap(\Sigma_c).$$

We next partition the surface $\Sigma \setminus c = \Sigma_c$ as

$$\Sigma_c = \Sigma_c^+ \sqcup \Sigma_c^-,$$

where each $\Sigma_c^\pm$ is a union of components of Σ_c, as follows. The subsurface Σ_c^- is the union of those components Σ_i of Σ_c such that the restriction map

$$A_P \to Map(\Sigma_i)$$

is the trivial homomorphism. In other words, a component Σ_j of Σ_c is contained in Σ_c^+ iff there exists $a \in A_P$ which restricts to a pseudo-Anosov element of $Map(\Sigma_j)$.

This partition of Σ_c is preserved by $N_{PMap(\Sigma)}(A_P)$ and we obtain

$$\pi = (\pi^+, \pi^-) : N_{PMap(\Sigma)}(A_P) \longrightarrow PMap(\Sigma_c^+) \times PMap(\Sigma_c^-) < PMap(\Sigma_c).$$

Clearly, the images $\pi^{\pm}(N_{PMap(\Sigma)}(A_P)) < PMap(\Sigma_c^{\pm})$ are contained in the normalizer

$$N_{PMap(\Sigma_c^{\pm})}(A_P^{\pm}), \quad \text{where} \quad A_P^{\pm} = \pi^{\pm}(A_P).$$

By Corollary 6.3, the group $Z_{PMap(\Sigma_c^+)}(A_P^+)$ is free abelian. Since A_P^- is trivial , $Z_{PMap(\Sigma_c^-)}(A_P^-) = PMap(\Sigma_c^-)$. We summarize these observations as

Lemma 6.6. For the groups $A_P^{\pm} = \pi^{\pm}(A_P)$, we have: $Z_{PMap(\Sigma_c^+)}(A_P^+) \cong \mathbb{Z}^r$ and $Z_{PMap(\Sigma_c^-)}(A_P^-) = PMap(\Sigma_c^-)$. Here $r = b_0(\Sigma_c^+)$.

7. Quasihomomorphisms to mapping class groups

In this section we will extend the rigidity results from $CAT(0)$ and hyperbolic target groups to mapping class groups. The main result of this section, a rigidity theorem for quasihomomorphisms to mapping class groups is similar to Theorem 5.5, except that the centralizers in mapping class groups do not (virtually) split.

Theorem 7.1. Suppose that Σ is an oriented connected surface of finite type and $f : G \to Map(\Sigma)$ is a quasihomomorphism. Then there exists a finite index subgroup $G^o < G$, a quasihomomorphism $f^o : G^o \to Map(\Sigma)$, $f^o \sim f$, such that:

1. $f^o(G^o) \subset PMap_c(\Sigma)$ for some (possibly empty) essential multiloop $c \subset \Sigma$.
2. The surface $\Sigma_c = \Sigma \setminus c$ admits a partition into subsurfaces $\Sigma_c = \Sigma_c^+ \sqcup \Sigma_c^-$, for which we have the exact sequence

$$1 \to T_c \to PMap_c(\Sigma) \stackrel{(\pi^+,\pi^-)}{\longrightarrow} PMap(\Sigma_c^+) \times PMap(\Sigma_c^-) \to 1,$$

as in §6.5.

3. The maps $f^{\pm} = \pi^{\pm} \circ f^o$ satisfy:

a. f^+ is a quasihomomorphism with free abelian target.

b. f^- is a homomorphism.

Proof. In what follows, we consider a quasihomomorphism $f : G \to Map(\Sigma)$ with infinite image. In view of Theorem 1.2, there exists a finite index subgroup $G_o < G$ and a quasihomomorphism $f_o : G_o \to Map(\Sigma)$, $f_o \sim f$, such that:

$$\Delta_{f_o} < Map(\Sigma)$$

is an abelian subgroup central in $\langle f_o(G_o) \rangle$. Consider the sequence

$$1 \to PMap(\Sigma) \to Map(\Sigma) \to Aut(H_1(\Sigma, \mathbb{Z}/3)) \to 1.$$

Applying Proposition 3.9 to f_o and this sequence, we replace G_o with its finite index subgroup $G^o := G_o'$ and replace f_o with a quasihomomorphism $f^o = f_o' : G^o \to PMap(\Sigma)$, $f^o \sim f_o$, such that

$$A := \Delta_{f^o} < \Delta_{f_o}$$

and $f^o(G^o)$ still centralizes A:

$$f^o : G^o \to Z_{PMap(\Sigma)}(A).$$

Since the image of f^o is contained in the pure mapping class group, the group $A = A_P$ is free abelian (of finite rank). If A is trivial, f^o is a homomorphism and we are done. Therefore, we will assume from now on that the group A is nontrivial.

So far, the proof is analogous to the one for CAT(0) groups. However, unlike in the $CAT(0)$ setting, centralizers in the mapping class group do not virtually split.

There are the following possibilities for the infinite group A (see §6.5):

1. Pseudo-Anosov case: There exists a pseudo-Anosov element $a \in A$. Then, the group $Z_{PMap(\Sigma)}(A)$ is infinite cyclic. It then follows that the quasihomomorphism $f^o : G^o \to PMap(\Sigma)$ has infinite cyclic image, which concludes the proof in this case.

2. Reducible case: A contains nontrivial reducible elements. As in §6.5, we have an A-invariant essential multiloop $c = c_A \subset \Sigma$, split the surface $\Sigma_c := \Sigma \setminus c$ as $\Sigma_c^+ \sqcup \Sigma_c^-$ and obtain homomorphisms

$$Z_{PMap(\Sigma)}(A) < PMap_c(\Sigma) \xrightarrow{\pi} PMap(\Sigma_c) = PMap(\Sigma_c^+) \times PMap(\Sigma_c^-),$$

$$\pi = (\pi^+, \pi^-), \quad \pi^\pm : PMap(\Sigma; c) \to PMap(\Sigma_c^\pm).$$

As we observed in Lemma 6.6, $\pi^+(Z_{PMap(\Sigma)}(A)) \cong \mathbb{Z}^r$, where r is the number of components of Σ_c^+. Therefore, for $A^+ = \pi^+(A)$, we obtain the quasihomomorphism

$$f^+ = \pi^+ \circ f^o : G^o \to Z_{PMap(\Sigma_c^+)}(A^+) \cong \mathbb{Z}^r.$$

As for Σ_c^-, the projection $\pi^-(A)$ is trivial and, since A contains the defect subgroup of f^o, the composition

$$f^- = \pi^- \circ f^o : G^o \to Z_{PMap(\Sigma_c^-)}(A^-) = PMap(\Sigma_c^-)$$

is a homomorphism. □

Corollary 7.2. Suppose that Γ is an irreducible lattice in a connected semisimple Lie group of rank ≥ 2, without compact factors. Then every quasihomomorphism of Γ to a mapping class group $Map(\Sigma)$ has finite image.

Proof. Suppose to the contrary that $f : \Gamma \to Map(\Sigma)$ is an unbounded quasihomomorphism. As in Theorem 7.1, we replace Γ with its finite index subgroup Γ^o (which is still an irreducible lattice of rank ≥ 2) and replace f with $f^o \sim f$, $f^o : \Gamma^o \to PMap(\Sigma)$. The compositions

$$f^\pm = \pi^\pm \circ f^o : \Gamma^o \to PMap(\Sigma_c^\pm),$$

satisfy the property that f^+ is a quasihomomorphism to a free abelian group A_1 and f^- is a homomorphism. The homomorphism f^- has to have finite image (see [2, 14, 22]); actually, in our setting, the image of f^- is trivial since $PMap(\Sigma_c^-)$ is torsion-free. Therefore, the image of the map f^o is contained in the abelian subgroup $B < PMap(\Sigma)$, the preimage $(\pi^+)^{-1}(A_1)$. Therefore, f^o is bounded in view of [7]. A contradiction. □

8. Quasihomomorphisms to groups acting trees

Suppose T is a simplicial tree and $H = Aut(T)$ is the group of automorphisms of T acting on T without inversions.

Definition 8.1. Suppose that $T' \subset T$ is a nonempty simplicial subtree and that $f : G \to Aut(T)$ is a quasihomomorphism whose image preserves T'. Let $H' = Aut_{T'}(T)$ denote the subgroup of $Aut(T)$ preserving T'. We have the restriction homomorphism $r : H' \to Aut(T')$. The composition $f' := r \circ f$ is a quasihomomorphism $f' : G \to Aut(T')$. In this situation we will say that the quasihomomorphism f is a *lift* of the quasihomomorphism f'.

We now proceed with the analysis of quasihomomorphisms $f : G \to H = Aut(T)$. Using Theorem 3.6, we find $f_o : G_o \to H_o = \langle f_o(G_o) \rangle$, such that $\Delta = \Delta_{f_o}$ is central in H_o.

Case 1. Axial case: Suppose that Δ contains an axial isometry δ of T, i.e., an isometry which preserves a complete geodesic T' in T and acts on T' as a nontrivial translation, i.e., T' is the *axis* of δ. Since each axial isometry has unique axis, the axis T' of δ is invariant under H_o and H_o acts on L by integer translations. (Centrality of Δ implies that every element of H_o preserves the orientation on T'.) Let

$$Aut^+_{T'}(T) < Aut_{T'}(T)$$

denote the subgroup of $Aut(T)$ preserving T' and its orientation. We have a natural homomorphism

$$\tau : Aut^+_{T'}(T) \to \mathbb{Z},$$

sending each $h \in Aut^+_{T'}(T)$ to the translation number for its action on T'. Composing f_o with τ we obtain a quasimorphism

$$f'_o = \tau \circ f_o : G_o \to \mathbb{Z}.$$

Thus, in this setting, f_o is a lift of a quasihomomorphism to $\mathbb{Z}$.

Case 2. Elliptic case: Suppose that Δ contains only elliptic isometries, i.e., each element of Δ has a fixed point in T. Recall that the defect group Δ is finitely generated abelian.

Lemma 8.2. Let A be a finitely generated abelian group acting isometrically on a tree T such that every element of A is elliptic. Then the fixed-point set of the action of A on T is nonempty.

Proof. We let $A_1, \ldots, A_n$ denote cyclic factors of A. The fixed subtree T_i of each A_i is nonempty. We claim that the tree

$$T' = T_1 \cap \ldots \cap T_n$$

is nonempty. The proof is by induction on n. The claim is clear for $n = 1$. Assume that it holds for $n - 1$. The subgroup $A' < A_1 \times \ldots \times A_{n-1} < A$ preserves the tree T_n and each element of A' acts on T_n as an elliptic isometry. Thus, the claim follows from the induction hypothesis. □

Applying this lemma to the group $A = \Delta_{f_o}$, we conclude that its fixed-point set in T is a nonempty subtree $T' \subset T$. By the normalization property, this subtree has to be invariant under H_o and, as above, we obtain the homomorphism

$$f'_o = r \circ f_o : G_o \to H' = Aut(T').$$

Hence, the quasihomomorphism f_o is a lift of the homomorphism f'_o.

This proves:

Lemma 8.3. If $f : G \to H = Aut(T)$ is a quasihomomorphism then, there exists a quasihomomorphism $f_o : G_o \to H$, $f_o \sim f$, such that:

1. Either f_o is a lift of a quasimorphism $f'_o : G_o \to \mathbb{Z} < H$, or
2. f_o is a lift of a homomorphism $f'_o : G_o \to H' = Aut(T')$ where $T' \subset T$ is a nonempty subtree.

Corollary 8.4. Suppose that G_o has no unbounded quasimorphisms and satisfies the property FA (e.g., G is an irreducible lattice in a connected semisimple Lie group of rank ≥ 2). Then there exists a subgroup $G^o < G_o$ of finite index and a quasihomomorphism $f^o : G^o \to Aut(T)$, $f^o \sim f_o$, such that $f^o(G^o)$ fixes a vertex in T.

Proof. Since G_o satisfies the property FA, $f_o(G)$ has a fixed vertex in T' in the *elliptic case*. Hence, in this situation, we can take $G^o = G_o, f^o = f_o$. Consider now the axial case. By the assumptions, the quasimorphism $f'_o : G_o \to \mathbb{Z}$ has finite image. Therefore, we apply Proposition 3.9 to the exact sequence

$$1 \to K \to Aut^+_{T'}(T) \xrightarrow{\tau} \mathbb{Z} \to 1$$

and conclude that there exists a finite index subgroup $G^o < G_o$ and a quasihomomorphism $f^o : G^o \to K$ with $f^o \sim f_o$. The image of f^o fixes each vertex of T'. □

Corollary 8.5. Suppose that H is the fundamental group of a graph of groups where every vertex group is hyperbolic. Then for every group G satisfying the hypothesis of Corollary 8.4, each quasihomomorphism $f : G \to H$ has finite image.

9. Other generalizations of homomorphisms

In this section we compare the notion of quasihomomorphisms used in this paper and going back to Ulam, with several other notions. In order to avoid the notation confusion, we will refer to quasihomomorphisms used earlier as *Ulam–quasihomomorphisms*. The other notions discussed in this section are equivalent to the one of Ulam–quasihomomorphism when the target is $\mathbb{Z}$, but differ in general.

9.1. **Algebraic and geometric quasihomomorphisms.** Let G and H be groups and d is a left-invariant metric on H. A map $f : G \to H$ is an *algebraic quasihomomorphism* if there exists a bounded subset $S \subset H$ such that for all $x, y \in G$ we have:

$$f(xy) = s_1 f(x) s_2 f(y) s_3, \quad s_i \in S, i = 1, 2, 3.$$

The true novelty in this definition (comparing to the one of Ulam–quasihomomorphisms) is presence of the element s_2. This class of maps is preserved by the following *bi-bounded perturbation* procedure: Pick a bounded subset $B \subset (H, d)$ and consider a map $f' : G \to H$ such that for each $x \in G$, $f'(x) \in Bf(x)B$. Then f' is again an algebraic quasihomomorphism.

Alternatively, one can require the more restrictive condition

$$f(xy) = f(x) s_2 f(y) s_3, \quad s_i \in S, i = 2, 3,$$

where S is a bounded subset of (H, d). We refer to such maps as *geometric quasihomomorphisms*. Geometric and algebraic quasihomomorphisms are stable under bounded perturbations. This presents a sharp contrast with Ulam's quasihomomorphisms (cf. Theorem 4.4).

We let $AQHom(G, (H, d))$ and $GQHom(G, (H, d))$ denote the sets of algebraic and geometric quasihomomorphisms, and denote by $UQHom(G, (H, d))$ the set of Ulam-quasihomomorphisms.

Example 9.1. 1. Each map $f : H \to H$ such that $\mathrm{dist}(f, id) < \infty$, is a geometric quasihomomorphism.

2. Compositions of algebraic (respectively, geometric) quasihomomorphisms are again (respectively, geometric) quasihomomorphisms.

We will give more interesting examples of geometric quasihomomorphisms in the next section.

A situation when geometric quasihomomorphisms appear naturally is the one of Margulis-type superrigidity: Suppose that $\Gamma < G$ is a uniform lattice in a connected Lie group (equipped with a left-invariant Riemannian metric) and $\phi : \Gamma \to (H, d)$ is a homomorphism. Then for a nearest-point projection $\rho : G \to \Gamma$ (which is a geometric quasihomomorphism), the composition

$$f = \phi \circ \rho : G \to (H, d)$$

is again a geometric quasihomomorphism. If G is a simple noncompact group of rank ≥ 2, then the Margulis Superrigidity Theorem implies that such geometric quasihomomorphism f is within finite distance from a homomorphism $G \to H$, provided that H is another connected Lie group (and d is induced by a left-invariant Riemannian metric on H). This leads to:

Question 9.2. Suppose that G is a connected simple Lie group of real rank ≥ 2 and (H, d) is a connected Lie group with trivial center, equipped with a metric d induced by a left-invariant Riemannian metric on H. Is it true that *every* geometric quasihomomorphism $f : G \to (H, d)$ is within finite distance from a homomorphism?

Note that the answer is clearly negative for all rank 1 Lie groups G, for instance, because these groups contain uniform lattices admitting unbounded quasimorphisms to $\mathbb{Z}$.

Problem 9.3. Describe $AQHom(G, H)$ for simple connected Lie groups G, H of rank ≥ 2. Is it true that each $f \in AQHom(G, G)$ is a bi-bounded perturbation of a homomorphism?

9.2. **Middle–quasihomomorphisms.** The following definition is inspired by a correspondence from Narutaka Ozawa.

Definition 9.4. A map $f : G \to H$ of two groups is a *middle–quasihomomorphism* if there exists a finite subset $S \subset H$ such that for all $x, y \in G$, there is $s \in S$ satisfying

$$f(xy) = f(x)sf(y).$$

We let $MQHom(G, H)$ denote the set of all middle–quasihomomorphisms $G \to H$.

By the definition, each middle–quasihomomorphism is geometric. As with other quasihomomorphisms, composition preserves middle–quasihomomorphisms.

Below is an interesting construction of *middle–quasihomomorphisms* $f : F_2 \to F_2$ which is a generalization of the Brooks' construction of quasimorphisms of free groups. Let a, b be free generators of the free group F_2. We say that two subwords q, q' of a reduced word w in the alphabet $a^{\pm 1}, b^{\pm 1}$ *intersect* if they contain a common nonempty subword, i.e. $q = q_1q_2q_3$, $q' = q'_1q'_2q'_3$ with q_2 nonempty and $q_2 = q'_2$. The subwords which do not intersect are called *disjoint*.

We say that two reduced words u, v in the alphabet $a^{\pm 1}, b^{\pm 1}$ are *totally nonoverlapping* if for every reduced word w in the alphabet $a^{\pm 1}, b^{\pm 1}$ any two subwords which are copies of distinct elements of

$$\{u, u^{-1}, v, v^{-1}\},$$

are disjoint. For instance, the words

$$u = a^m b a^m, \quad v = b^m a b^m, \quad m \geq 2, \tag{14}$$

satisfy this condition.

We now fix two two nonempty cyclically reduced totally nonoverlapping words u, v and set

$$T := \{u, u^{-1}, v, v^{-1}\}.$$

Let L denote the maximum of lengths of u and v. Since u and v are cyclically reduced, the biinfinite paths

$$\ldots uuu \ldots, \quad \ldots vvv \ldots$$

are invariant geodesics for u and v respectively in the Cayley graph of F_2 with respect to the generating set $\{a, b\}$. Since the words u, v are totally nonoverlapping, these invariant geodesics have finite intersection. In particular, the subgroup $H \leq F_2$ generated by u and v is free of rank 2 (with the generators u, v), since H cannot be cyclic.

Given a reduced word w in the alphabet $a^{\pm 1}, b^{\pm 1}$, consider all the subwords $t_1, \ldots, t_n$ (listed in the order of their appearance in w) which belong to the set T. Define the map

$$f : F_2 \to H,$$
$$f(w) = f_{u,v}(w) := t_1 \ldots t_n \in F_2.$$

If $n = 0$, we set $f(w) = 1$. Let $\alpha : H \to \mathbb{Z}$ denote the homomorphism sending v to $0 \in \mathbb{Z}$ and u to $1 \in \mathbb{Z}$. Then the composition $\beta = \alpha \circ f$ is the Brooks quasimorphism $F_2 \to \mathbb{Z}$, associated with the word u, see [5]. It is clear from the construction that $f(w^{-1}) = (f(w))^{-1}$ for each $w \in F_2$.

Example 9.5. Let $u = a^a b a^2, v = b^2 a b^2$. Then for

$$w = aabaabaabbabbaa$$

we have

$$f_{u,v}(w) = uuv.$$

Theorem 9.6. Assume that u, v are cyclically reduced totally nonoverlapping words u, v, such that, regarded as elements of F_2, u and v do not belong to the cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$. (For instance, we can take u and v as in (14).) Then:

1. f is a middle–quasihomomorphism.
2. The image of f is infinite and is not contained in the R-neighborhood of an infinite cyclic subgroup of F_2 for any $R < \infty$.
3. The map f is not within finite distance from a homomorphism.

Proof. 1. We first check that f is a middle–quasihomomorphism. Consider two reduced words w_1, w_2, which are (reduced) products

$$w_1 = w_1' w_1'', w_2 = w_2' w_2'',$$

where w_1'', w_2' are maximal with the property that in the group F_2,

$$w_1'' w_2' = 1.$$

We let $J(w_i)$ denote the ordered set (listed in the order of their appearance in w_i) of subwords in w_i which are copies of elements of T intersecting both w_i', w_i''.

Remark 9.7. Note that $J(w_i)$ need not be a singleton as copies of, say, u, appearing in w_i can overlap. For instance, for $u = a^2ba^2$ and $w_1' = aaba, w_1'' = abaa$, we have

$$J(w_1) = (u, u).$$

However, due to the "totally nonoverlapping" condition, each $J(w_i)$ consists only of copies of u, or of u^{-1}, or of v or of v^{-1}.

Then the ordered product Y_i of the elements of $J(w_i)$ has length $\le L^2$. Furthermore,

$$f(w_1) = X_1Y_1Z_1, \quad f(w_2) = Z_1^{-1}Y_2Z_2,$$

and for the element $w_3 \in F_2$ represented by w_1w_2 we have

$$f(w_3) = X_1Y_3Z_2,$$

where $|Y_3| \le L^2$. Set

$$s_2 = Y_1^{-1}Y_3Y_2^{-1}.$$

Then

$$f(w_3) = f(w_1)s_2f(w_2),$$

where s_2 has length $\le 3L^2$. This proves the first claim.

2. It is clear that $f(u^n) = u^n$ and $f(v^n) = v^n$ for each n. Since the cyclic subgroups of F_2 generated by u and by v are not Hausdorff-close, the second claim of the theorem follows.

3. Since both u and v (regarded as elements of F_2) do not belong to the cyclic subgroups $\langle a\rangle, \langle b\rangle$, the words a^m, b^n, $m, n \in \mathbb{Z}$, contain no subwords from T. Therefore, the map f sends both cyclic subgroups $\langle a\rangle$ and $\langle b\rangle$ to $\{1\}$. It follows that for each map $f' : F_2 \to F_2$ within finite distance from f, the images of $\langle a\rangle$ and $\langle b\rangle$ are bounded. Hence, f' can be a homomorphism only if it is the constant map $F_2 \to \{1\}$. Since f is unbounded, we conclude that it cannot be within finite distance from a homomorphism. □

9.3. **Quasimorphisms of Hartnick and Schweitzer.** In their paper [18], which appeared shortly after the initial version of our paper was posted, Hartnick and Schweitzer introduce the following notion, which we will refer to as an HS–quasimorphism:

Definition 9.8. A map $f : G \to H$ of two groups is an HS–quasimorphism if for each quasimorphism $\varphi : H \to \mathbb{R}$, the composition $\varphi \circ f : G \to \mathbb{R}$ is a quasimorphism. (Note that H need not be equipped with a metric.) We will use the notation $HSQMor(G, H)$ for the set of HS–quasimorphisms.

In other words, Hartnick and Schweitzer take the concept of quasimorphisms (quasihomomorphisms to $\mathbb{R}$) as central, and then define HS–quasimorphisms in a categorical fashion. It is immediate that composition preserves HS–quasihomomorphisms. If we equip the target group H with a discrete proper left-invariant metric (whose choice is irrelevant and will be suppressed), then, clearly,

$$UQHom(G, H) \subset GQHom(G, H) \subset AQHom(G, H) \subset HSQMor(G, H),$$

$$MQHom(G,H) \subset GQHom(G,H) \subset AQHom(G,H) \subset HSQMor(G,H).$$

In particular, as with algebraic quasihomomorphisms, if $f_1 : G \to H$ is an HS–quasihomomorphism and $\text{dist}(f_1, f_2) < \infty$, then $f_2 : G \to H$ is again an HS–quasihomomorphism. Hartnick and Schweitzer prove, among other interesting results, that free groups F_n of finite rank $n \geq 2$ have abundant supply of HS–automorphisms. More precisely, let $QAut(F_n)$ denote the space of HS–quasiautomorphism $F_n \to F_n$, $Hom(F_n, \mathbb{R})$ is the space usual homomorphisms and $\mathcal{H}(F_n)$ the space of homogeneous quasimorphisms $F_n \to \mathbb{R}$. Then, according to Theorem 1 of [18], the closure of the linear span of the $QAut(F_n)$-orbit of $Hom(F_n, \mathbb{R})$ is the entire space $\mathcal{H}(F_n)$.

A drawback of Definition 9.8 is that it is only meaningful for maps to groups H which admit abundant supply of quasimorphisms, e.g., hyperbolic groups. If H is an irreducible lattice of rank ≥ 2, then *every map* $G \to H$ is an HS–quasimorphism, as H has only bounded quasimorphisms. In contrast, Theorem 5.8 shows that if $\Gamma < G$ is an irreducible lattice in a connected semisimple Lie group G of rank ≥ 2, without nontrivial compact normal subgroups, then each Ulam-quasihomomorphism $f : \Gamma \to \Gamma$ has finite image or is an automorphism.

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