Rado's paracompactness theorem for conformal manifolds

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August 4, 2025

In this note, an (n-dimensional) manifold is a Hausdorff topological space equipped with a maximal smooth atlas with values in $\mathbb{R}^n$, no paracompactness is assumed. Recall that a *conformal structure* on a manifold M is a reduction of the structure group of TM from $GL(n, \mathbb{R})$ to $CO(n) = \mathbb{R}_+ \times O(n)$. Equivalently, a conformal structure is a collection of locally defined Riemannian metrics on M which are conformal to each other. We will refer to such locally defined Riemannian metrics as *local conformal metrics* on M. Once we know that M is paracompact, the structure group can be further reduced to $O(n)$ and local conformal metrics can be replaced by a single Riemannian metric on the entire M inducing the conformal structure. A *conformal manifold* is a manifold equipped with a conformal structure. We will suppress the notation for a conformal structure and denote conformal manifolds by a single letter, e.g., M. We will prove:

Theorem 1. *Every conformal manifold M of dimension $n \geq 2$ is paracompact.*[1]

This result is known as *Rado's theorem* for $n = 2$ and oriented conformal manifolds, equivalently, for Riemann surfaces (see e.g. [For81, §23] for a proof using Perron's method). It is well-known that paracompactness fails for complex manifolds of complex dimension ≥ 2 (examples are due to Calabi and Rosenlicht, [CR53]). The goal of this note is to show that Rado's theorem is actually a theorem of conformal, rather than complex, geometry.

Question 2. *What are other differential-geometric structures on manifolds which imply paracompactness?*

For instance, the existence of a symplectic structure is not an obstruction to paracompactness, as one can take, for instance, the canonical symplectic form on the cotangent bundle of a non-paracompact manifold.

Acknowledgements. This work was motivated by a discussion of Perron's method and Rado's theorem with Bernhard Leeb.

[1] Note that this result obviously fails for 1-dimensional manifolds.

1 Topological preliminaries

A collection $\mathcal{C}$ of subsets of a topological space X is said to be *locally finite* if every $x \in X$ has a neighborhood which intersects only finitely many members of $\mathcal{C}$. A topological space X is called *paracompact* if every open cover of X admits a locally finite open refinement. (Unlike in the definition of compactness, a *refinement* cannot be replaced by a *subcover*.) This notion was introduced by Dieudonné in [Die44].

Theorem 3. *(See e.g. [Eng89, Theorem 5.1.3].) Every metrizable topological space is paracompact.*

This theorem has a "converse" of sorts:

Theorem 4. *(Smirnov's theorem, see e.g. [Eng89, 5.4.A].) Every locally metrizable paracompact space is metrizable.*

In general, paracompactness is not a hereditary property. However:

Lemma 5. *Let X be a locally metrizable space (e.g. a manifold). Then every paracompact subset Y of X is hereditarily paracompact.*

Proof. Since X is locally metrizable, so is Y, hence, by Smirnov's theorem, Y is metrizable. This implies that every subset $Z \subset Y$ is also metrizable, hence, paracompact. □

Theorem 6. *(See e.g. [Eng89, Theorem 5.1.34].) Suppose that X is a topological space which is a union of a locally finite family of closed paracompact subsets. Then X is itself paracompact.*

Corollary 7. *If X is a union of finitely many closed paracompact subsets, then X is paracompact.*

Recall that a space is said to be σ-compact if it is a union of countably many compact subsets. The following theorem was first proven by Dieudonné in [Die44]. For a textbook reference, see [Bou89, Ch. I.10, Theorem 5].

Theorem 8. *Suppose that X is a locally compact Hausdorff σ-compact space. Then X is paracompact. Conversely, a locally compact Hausdorff space X is paracompact if and only if it is the coproduct of a family of (pairwise disjoint) subspaces $X_i, i \in J$, each of which is Hausdorff, locally compact and σ-compact.*

Corollary 9. *A manifold is paracompact if and only if every connected component is second countable.*

Lemma 10. *Every compact subset K of a connected manifold M is contained in an open (in M) connected relatively compact (hence, paracompact) submanifold $N \subset M$.*

Proof. First, let $\mathcal{U}$ be a finite cover of K by open coordinate balls $B_i \subset M$. Let $\bar{B}_i$ denote the corresponding closed balls. Then

$$\bigcup_i \bar{B}_i$$

is a compact subset of M with finitely many connected components. Connecting these components by finitely many paths we get a compact connected subset $L \subset M$ containing K. Let $\mathcal{V}$ be a finite cover of L by open coordinate balls. Then the union N of these balls is paracompact, connected and contains K. □

Even though the manifold N in this lemma is not canonical, we will use the notation $\hat{K}$ for N.

Corollary 11. *Suppose that M is a connected manifold and $K_i \subset M, i \in \mathbb{N}$, is a countable collection of compact subsets. Then $\bigcup_i K_i$ is contained in an open paracompact connected subset $N \subset M$.*

Proof. Without loss of generality, we may assume that $K_i \subset K_{i+1}$ for every i. Consider the family $\{\hat{K}_i : i \in \mathbb{N}\}$. The union N of these open connected subsets is again open and connected. At the same time, it is contained in

$$Y := \bigcup_{i \in \mathbb{N}} cl(\hat{K}_i),$$

a σ-compact subset of M. Then Y is paracompact by Proposition 8. Therefore, $N \subset Y$ is paracompact as well. □

2 Proof of Theorem 1

In order to prove Theorem 1 it suffices to consider connected conformal manifolds and prove that they are always metrizable. More precisely, we will check that the complement to a closed coordinate ball in M admits a conformally-natural metric. The construction of this metric mostly follows the work of Ferrand, [Fer96]. Hidden behind the construction is again Perron's method, but it is used differently from the standard proofs of Rado's theorem for Riemann surfaces. We let $Lip(M)$ denote the space of Lipschitz continuous functions on M. (Ferrand uses a slightly different functional space.) Lipschitz continuous functions are differentiable a.e. on M and norms of their gradients are locally bounded and, hence, locally integrable. We let $Lip_p(M)$ denote the subspace consisting of functions in $Lip(M)$ with *paracompact support* and $Lip_c(M)$ the subspace of $Lip_p(M)$ consisting of functions with compact support. We will use the fact that for $f_1, f_2 \in Lip(M)$, $\max(f_1, f_2)$ is also in $Lip(M)$.

Recall that our manifold M is n-dimensional. Given a function $f \in Lip(M)$ we define its *energy-density* $e_f(x)$ as

$$|\nabla f(x)|^n dV, x \in M,$$

where the gradient, its norm and the volume density dV are defined with respect to a local conformal metric (defined in a neighborhood of x). The definition of e_f is independent of the choice of a local conformal metric and e_f is locally integrable. Assuming that f is in $Lip_p(M)$, we have the *n-energy integral*

$$I_M(f) := \int_M e_f \in [0, \infty].$$

Note that this integral is well-defined since the support set of e_f is paracompact and, hence, we can use a partition of unity on this support to define the integral.

Let C_1, C_2 be two closed subsets of M. The *capacity of the pair* (C_1, C_2) is defined as

$$Cap_M(C_1, C_2) = \inf_{f \in A(M, C_1, C_2)} I_M(f),$$

where $A(M, C_1, C_2)$, the space of *admissible functions* with respect to (C_1, C_2), consisting of functions $f \in Lip_p(M)$ such that $0 \leq f \leq 1$, $f|_{C_1} \equiv 0$, $f|_{C_2} \equiv 1$. Thus, $0 \leq Cap_M(C_1, C_2) = Cap_M(C_2, C_1) \leq \infty$. By the definition, capacity is a conformal invariant. It is also clear that if N is an open subset of a conformal manifold M (with the induced conformal structure), then for any pair of closed subsets $C_1, C_2 \subset M$, we have

$$Cap_N(C_1, C_2) \leq Cap_M(C_1, C_2) \tag{1}$$

as the restriction map sends $A(M, C_1, C_2)$ to $A(N, C_1, C_2)$ and decreases the energy. Recall that a compact metrizable space is called a *continuum* if it is connected. A *nondegenerate continuum* is one which has cardinality at least two (hence, cardinality of continuum). A compact subset of a manifold is metrizable, hence, every compact connected subset of a manifold is a continuum. We will need the following result from [Fer96, (1.3)]:

Lemma 12. *Let $B = B(\mathbf{0}, 1) \subset \mathbb{R}^n$ be the open unit ball equipped with some conformal structure (not necessarily the standard one). Then for every $\epsilon > 0$ and $r > 0$ there exists $f \in Lip_c(B)$ which is identically equal to 1 on the ball $B(\mathbf{0}, r)$ and satisfies $I_B(f) < \epsilon$.*

From now on, we will assume that our manifold M is connected.

Lemma 13. *1. Suppose that C_1, C_2 are disjoint compact subsets of M. Then $Cap(C_1, C_2) < \infty$.*

2. Suppose that C_1, C_2 are nondegenerate continua in M. Then $Cap_M(C_1, C_2) > 0$.

Proof. 1. Take an open paracompact subset $U \subset M$ containing $C_1 \cup C_2$ and a compactly supported smooth function $f \in A(U, C_1, C_2)$. Then extending f by 0 to the rest of M, we obtain an admissible function of finite energy.

2. Let $N = \hat{C} \subset M$ be an open connected paracompact subset containing $C = C_1 \cup C_2$. Thus, $Cap_N(C_1, C_2) \leq Cap_M(C_1, C_2)$. But for connected paracompact conformal manifolds N positivity of $Cap_N(C_1, C_2)$ is proven in [Fer96, (3.6)]. Now the conclusion follows from the inequality (1). □

Following Ferrand, [Fer96], we next define a certain pseudometric on M associated canonically with the conformal structure of M. For a compact $K \subset M$ set

$$Cap_M(K) := \inf_f I_M(f),$$

where the infimum is taken over all functions $f \in Lip_c(M)$ which equal to 1 on K. Such functions will be called K-*admissible*. Clearly, if $N \subset M$ is an open connected subset, then for every compact $K \subset M$,

$$Cap_M(K) \leq Cap_N(K).$$

Lemma 14. *$Cap_M(K) < \infty$ for every compact $K \subset M$.*

Proof. Take a relatively compact open subset $N \subset M$ containing K. Then pick a function $f \in C^1_c(N)$ which is identically 1 on K and extend it by zero to the rest of M. Thus, $Cap_M(K) \leq I_M(f) < \infty$. □

Lemma 15. *If $K_1 \subset K_2$ are compacts in M, then*

1. *$Cap_M(K_1) \leq Cap_M(K_2)$,*
2. *$Cap_M(K_1 \cup K_2) \leq Cap_M(K_1) + Cap_M(K_2)$.*

Proof. The first part is immediate. To prove the second part, take K_i-admissible functions $f_i, i = 1, 2$, on M and set $f := \max(f_1, f_2)$. Then f is K-admissible for $K = K_1 \cup K_2$. Set

$$A_1 := \{x : f_1(x) > f_2(x)\}, A_2 := \{x : f_2(x) > f_1(x)\}, A_0 := \{x : f_1(x) = f_2(x)\}.$$

Thus, $M = A_1 \sqcup A_2 \sqcup A_0$. We have

$$I_M(f) = I_{A_1}(f_1) + I_{A_2}(f_2) + \int_{A_0} e_f.$$

Clearly,

$$I_{A_1}(f_1) + I_{A_2}(f_2) = \int_{A_1 \cup A_2} (e_{f_1} + e_{f_2}) = \int_{A_1 \cup A_2} e_f.$$

It remains to analyze the integrals of energy-densities over A_0 (note that A_0 can have positive measure). Take a point $x_0 \in A_0$, and fix a local conformal metric g on an open coordinate ball $B \subset M$ centered at x_0. Then, with respect to this metric,

$$|\nabla f(x)| \leq \max(|\nabla f_1(x)|, |\nabla f_2(x)|) \leq |\nabla f_1(x)| + |\nabla f_2(x)|, x \in B \cap A_0.$$

Hence (denoting dV the volume density of g),

$$\int_{B\cap A_0} e_f \le \int_{B\cap A_0} (|\nabla f_1(x)| + |\nabla f_2(x)|)dV = \int_{B\cap A_0} (e_{f_1} + e_{f_2}).$$

Therefore,

$$\int_{A_0} e_f \le \int_{A_0} (e_{f_1} + e_{f_2}).$$

Lemma follows. □

Lemma 16. *Let $N \subset M$ be the complement to a closed coordinate ball $\bar{B} \subset M$. Then for every nondegenerate continuum $K \subset N$ we have $Cap_N(K) > 0$.*

Proof. Take a K-admissible function $f \in Lip_c(N)$ and extend it by 0 to $\bar{B}$. We get a function $u \in A(M, \bar{B}, K)$ and $I_N(f) = I_M(u)$. By Lemma 13 (Part 2), there exists $r > 0$ independent of f, such that $I_M(u) \ge r$. Hence, $I_N(f) \ge r > 0$ and lemma follows. □

Lemma 17. *For every singleton $K = \{x\} \subset M$, $Cap_M(K) = 0$.*

Proof. Let B be an open coordinate ball in M centered at x. Then, according to Lemma 12, there exists a sequence of smooth functions $f_i \in Lip_c(B)$ which are all equal to 1 at x and

$$\lim_{i\to\infty} I_B(f_i) = 0.$$

Extending functions f_i by 0 to the rest of M, we conclude that $Cap_M(K) = 0$. □

Definition 18. *For points $x, y \in M$ define $\mu_M(x, y) := \inf_C Cap_M(C)$, where the infimum is taken over all continua $C \subset M$ containing $\{x, y\}$.*

Lemma 19. *The function μ_M is a finite pseudometric on M.*

Proof. Symmetry of μ_M is clear. Since M is a connected manifold, it is path-connected; hence, every two points $x, y \in M$ belong to a continuum $C \subset M$. Lemma 14 then implies that $\mu_M(x, y) \le Cap_M(C) < \infty$, hence, μ_M is finite. The triangle inequality follows from Lemma 15 (Part 2). Lemma 17 implies that $\mu_M(x, x) = 0$, since we can take $C = \{x\}$ as our continuum containing $\{x\}$. □

Definition 20. *A conformal manifold M is said to be of* Class I *if the pseudometric μ_M is not a metric and is said to be of* Class II *otherwise.*

Proposition 21. *The following are equivalent:*

1. *For some pair of distinct points $x, y \in M$, $\mu_M(x, y) = 0$, i.e. M is of Class I.*
2. *For all pairs of points $x, y \in M$, $\mu_M(x, y) = 0$.*
3. *For every continuum $C \subset M$, $Cap_M(C) = 0$.*
4. *There exists a nondegenerate continuum $C \subset M$, such that $Cap_M(C) = 0$.*

Proof. The only part which is not obvious is the implication (1)⇒(3). This implication is proven in [Fer96, (6.8)] for paracompact manifolds. One way to argue would be to adapt her proof to the general case where manifolds are not assumed to be paracompact. Instead, we reduce the general case to the paracompact one. Since $\mu_M(x,y) = 0$, there exists a sequence of nondegenerate continua $C_i \subset M$ (containing $\{x,y\}$) and C_i-admissible functions $f_i \in Lip_c(M)$ such that $I_M(f_i) < 1/i$. Let K_i denote the (compact) support set of f_i, hence, $C_i \subset K_i$. The union

$$C \cup \bigcup_i K_i$$

is contained in a paracompact open connected subset $N \subset M$, see Corollary 11. Thus, $\mu_N(x,y) = 0$ (since each f_i restricts to a compactly supported function on N). But this implies that $Cap_N(C) = 0$ as proven by Ferrand, [Fer96, (6.8)]. Since $Cap_M(C) \leq Cap_N(C) = 0$, we conclude that $Cap_M(C) = 0$ as well. □

Corollary 22. *(See [Fer96, Example 6.9(b)].) Let $K = \bar{B} \subset M$ be a closed coordinate ball. Then the manifold $N := M - K$ is of Class II.*

Proof. So far, we have not used the assumption that M has dimension > 1 (except, indirectly, in Lemma 13). We will use it now explicitly and observe that due to this dimension assumption, the manifold N is connected. Suppose that μ_N is not a metric. Then, by Proposition 21, there exists a nondegenerate continuum $C \subset N$ such that $Cap_N(C) = 0$, which implies that $Cap_M(C, K) = 0$. But this contradicts Part 2 of Lemma 13. □

Proposition 23. *If M is of Class II, then the metric μ_M metrizes M as a topological space.*

Proof. 1. We first prove that the function $\mu_M : M^2 \to \mathbb{R}$ is continuous at the diagonal, i.e. if x_i, y_i are sequences in M converging to the same point $z \in M$, then $\mu_M(x_i, y_i) \to 0$. Fix an open coordinate ball $B \subset M$ centered at z. Without loss of generality, $x_i, y_i \in B$ for all i. Then,

$$\mu_B(x_i, y_i) \to 0,$$

see Lemma 12. Since $\mu_M(x_i, y_i) \leq \mu_B(x_i, y_i)$, we get

$$\lim_{i\to\infty} \mu_M(x_i, y_i) = 0 = \mu_M(z, z).$$

2. Let us check continuity of μ_M at general pairs $(x,y) \in M^2$. Consider sequences $x_i \to x, y_i \to y$ in M. Then, by the triangle inequality for μ_M and Part 1 of the proof:

$$\mu_M(x,y) \leq \liminf_{i\to\infty}(\mu_M(x,x_i) + \mu_M(x_i,y_i) + \mu_M(y_i,y)) \leq \liminf_{i\to\infty} \mu_M(x_i,y_i).$$

Similarly,

$$\limsup_{i\to\infty} \mu_M(x_i, y_i) \leq \mu_M(x,y).$$

It follows that μ_M is continuous at (x, y). Thus, the manifold topology of M is stronger than the metric topology induced by μ_M.

3. Let us prove that the metric topology is stronger than the manifold topology. Let $B \subset M$ be an open coordinate ball centered at $x \in M$. We will show that there exists $r > 0$ such that for every $z \in M - B$, $\mu_M(x, z) \geq r$. Indeed, since the function $h(y) = \mu_M(x, y)$ is continuous and $S = \partial B$ is compact, the function h has positive minimum $r > 0$ on S. Take $z \in M - B$ and a continuum $C \subset M$ connecting x and z. This continuum has to intersect S at some point y. Therefore, $r \leq \mu(x, y) \leq Cap_M(C)$ and, thus, $\mu_M(x, z) \geq r$. Proposition follows. □

We can now finish the proof of the theorem. Consider a closed coordinate ball $\bar{B} \subset M$ which we will identify with the closed unit ball $\bar{B}(\mathbf{0}, 1) \subset \mathbb{R}^n$. Take two smaller closed balls,

$$\bar{B}_1 = \bar{B}(\mathbf{0}, 1/4) \subset \bar{B}_2 = \bar{B}(\mathbf{0}, 1/2) \subset \bar{B}(\mathbf{0}, 1) = \bar{B}.$$

By Corollary 22, the complement $N = M - \bar{B}_1$ is metrizable (by the metric μ_N). Let B_2 be the interior of $\bar{B}_2$; it is an open coordinate ball in M. Hence, the closed subset $X = M - B_2 \subset M$ is metrizable as well (by the restriction of the metric μ_N). Therefore, M is the union of two closed paracompact subsets: X and $\bar{B}$ (the latter is even compact). But a topological space which is the union of finitely many closed paracompact subsets is itself paracompact. Paracompactness of M follows.

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