COARSE FIBRATIONS AND A GENERALIZATION OF THE SEIFERT FIBERED SPACE CONJECTURE

MICHAEL KAPOVICH AND BRUCE KLEINER

Abstract. We prove a version of the Seifert fibered space conjecture for coarsely fibered 3-manifold groups.

1. Introduction

The main goal of this paper is to establish a coarse analogue *Seifert fibered space conjecture* for 3-manifolds. We recall that this conjecture (proved as the result of collective efforts of a large number of mathematicians in early 1990-s) asserts that:

If M is a 3-dimensional irreducible manifold whose fundamental group has infinite center, then M is a Seifert manifold.

Note, that if γ is a smooth loop in M representing a nontrivial element of the center of $G = \pi_1(M)$, then the inverse image of γ in the universal cover X of M is a family of proper (noncompact) curves L (not necessarily disjoint or embedded) which is invariant under the action of the fundamental group G. The above curves are not just proper in X, but are *uniformly proper*, i.e., there exists a *distortion function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each curve L,

$$\eta(d_L(x, y)) \le d(x, y)$$

where $\lim_{t\to\infty} \eta(t) = \infty$ and d_L is the distance measured along L and d is the distance in X with respect to a G-invariant Riemannian metric. Moreover, the distance between curves L, L' in this family cannot oscillate arbitrarily: if $x \in L, x' \in L'$ are within distance $\le r$ then any point $y \in L$ is within distance $\le \psi(r)$ from a point in L', where the function ψ does not depend on L, L', x, x', and y.

Similar examples of families of curves in Riemannian manifolds X appear as fibers of Riemannian submersions. One such example is given by Solv-manifolds:

Let $h : T^2 \to T^2$ be an Anosov affine mapping; recall there are two h-invariant foliations $\mathcal{A}$ of T^2 by affine geodesics. Let M be the mapping torus of h. Then each foliation $\mathcal{A}$ yields a 1-dimensional lamination $\mathcal{L}$ on M, whose lift to the universal cover X gives rise to a family of curves L satisfying all the above properties.

Note however that in the latter example, no curve L is invariant under a nontrivial element of G. Moreover, no curve L is Hausdorff-close to a curve invariant under

Date: September 17, 2004.

$g \in G \setminus \{1\}$. More examples of the same kind are given by irrational affine foliations on 3-dimensional tori. Combining the above examples we arrive to the following

Definition 1.1. Let X be a metric cell complex, see section 2 for the definition. For the purposes of the introduction the reader can assume that X is a simplicial complex, which admits a cocompact group action, where X is metrized so that each simplex is isometric to the standard simplex in the Euclidean space. We define a *coarse fibration* $\mathcal{F}$ of X as follows. Let $\{F_\alpha, \alpha \in A\}$ be a collection of finite-dimensional metric cell complexes ("fibers"); suppose that for each $\alpha \in A$ we are given an L-Lipschitz map $f_\alpha : F_\alpha \to \bar{F}_\alpha \subset X$ which is uniformly proper with the distortion function η independent on α. In addition we assume that

(1) The union $\cup_{\alpha \in A} \bar{F}_\alpha$ is Hausdorff-close to X, and

(2) There is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that if for $\alpha, \beta \in A$ there are points $x \in \bar{F}_\alpha, y \in \bar{F}_\beta$ such that $d(x, y) \leq r$ then the Hausdorff distance between $\bar{F}_\alpha, \bar{F}_\beta$ is at most $\psi(r)$.

If $G \curvearrowright X$ is an (isometric) group action then we say that a coarse fibration $\mathcal{F}$ is G-invariant if for each $g \in G$ and $\alpha \in A$ we have: $g \circ f_\alpha \in \mathcal{F}$. We will say that $\mathcal{F}$ is a coarse fibration by lines if F_α is isometric to $\mathbb{R}$ for each $\alpha \in A$.

Example 1.2. Suppose that X is a Cayley graph of a group G, $C \subset G$ is a central subgroup. Let F be a Cayley graph of C. There is a natural map $F \to \bar{F} \subset X$. Thus for each element $g \in G$ we have a map

$$f_g : F \to \overline{F_g} = g\bar{F} \subset X$$

induced by $C \subset gC \subset G$. Then the collection $\{(F, f_g), g \in G\}$ is a coarse fibration of X.

Observe that a coarse fibration is a coarse analogue of a Riemannian submersion between complete Riemannian manifolds: fibers of such submersion are equidistant from each other.

We note that coarse fibrations of the universal covers of 3-manifolds M appear in the recent work of D. Calegari [3]. Namely, he considers *quasi-homomorphisms* h of the fundamental group $G = \pi_1(M)$ into $\mathbb{R}$. The inverse images $h^{-1}(x), x \in \mathbb{R}$, define a coarse fibration of the group G and, therefore, of the universal cover X of M. Note however that such coarse fibrations are far from being *by lines*; rather, under an appropriate finiteness assumption, they are coarse fibrations by *quasi-planes*. Another instance where coarse fibrations appear in the geometric group theory is in relation to Bieri-Neumann-Strebel invariant for finitely-generated groups [1].

The main result of our paper is:

Theorem 1.3. *Suppose that M is a closed 3-manifold whose universal cover admits a $\pi_1(M)$–invariant coarse fibration by lines. Then either M is homotopy-equivalent to a Seifert manifold or to a* Solv*-manifold.*

In section 7.2 we will also classify all invariant coarse fibrations of universal covers of Seifert and Sol-manifolds.

As an application of our results we get an alternative proof of

Theorem 1.4. *(P. Tukia, G. Mess, D. Gabai, A. Casson and D. Jungreis.) Let M be a closed irreducible 3-manifold whose fundamental group has a nontrivial center. Then M is homeomorphic to a Seifert manifold.*

To be more precise, we are giving an alternative proof to the part of the above theorem which was proven by G. Mess in [14]. Our proof still relies upon the classification of uniform convergence groups acting on S^1, which was proven by Tukia, Gabai, Casson and Jungreis. We note that other alternative proofs of Geoff Mess' theorem were recently given by B. Bowditch [2] and S. Maillot [13].

Theorem 1.3 is used in our forthcoming work [8] to settle (under very strong extra assumptions) the *weak hyperbolization conjecture* for 3-dimensional manifolds:

The fundamental group of a compact 3-manifold M is either Gromov-hyperbolic or contains a rank 2 abelian subgroup.

A coarse fibration on the universal cover X of M in [8] appears as a family of intersection curves between leaves of certain immersed 2-dimensional laminations in X.

This paper is organized as follows.

In section 2 we review basics of the geometric group theory and controlled topology.

In section 3 we introduce coarse fibrations and discuss their basic properties. We define metric cell complexes which serve as *coarse bases* of coarse fibrations. In the case of a coarse fibration as in Example 1.2, the base is quasi-isometric to the quotient group G/C.

In section 4 we show that coarse fibrations (under appropriate uniform contractibility assumptions on the total space and the coarse fibers) behave homotopy-theoretically as products of the coarse base and a coarse fiber.

In section 5 we introduce the concept of *coarse Poincaré duality* and review some basic properties of metric cell complexes which satisfy this duality, this subject was discussed in much greater details in our previous paper [7]. For metric cell complexes which satisfy coarse Poincaré duality one proves coarse analogues of Jordan separation; for instance, if L is a geodesic in $Y^{(1)}$, where Y satisfies 2-dimensional coarse Poincaré duality, then some neighborhood of L separates $Y^{(1)}$ into exactly two components none of which is within finite distance from L. A similar property holds if instead of a geodesic one consider a *fat geodesic triangle* T in $Y^{(1)}$: one can define coarse interior and exterior of T.

We then prove one of the critical results of this paper: If X is a coarsely fibered uniformly contractible metric cell complex which is homeomorphic to an n-manifold, so that the fibers have locally compact cohomology of $\mathbb{R}^i$, then the coarse base Y of the fibration satisfies $n-i$-dimensional coarse Poincaré duality. The reader familiar with the paper of G. Mess [14] will note that this part of the paper is parallel to the part of [14], where it is proven that the group $\pi_1(M^3)/C$ is quasi-isometric to a planar surface, where C is an infinite cyclic central subgroup of $\pi_1(M^3)$. Although, 2-dimensional coarse Poincaré duality[1] for a metric cell complex Y *a priori* is not the same as Y being quasi-isometric to a planar surface, *coarse separation properties* of the coarse base Y suffice for our arguments. According to the results of our paper [10] we have the following dichotomy:

1. Either Y is Gromov-hyperbolic, with topological circle as the ideal boundary. In this case the quasi-action $G \curvearrowright Y^0$ is quasi-isometrically conjugate to an isometric action $G \curvearrowright \mathbb{H}^2$.

2. Or Y has polynomial growth and, in case Y is quasi-isometric to a finitely generated group Q, the group Q is virtually abelian of rank 2.

To put this into prospective, suppose that Y appears as a base of a coarse fibration $\mathcal{F}$ of a contractible 3-manifold X, so that $\mathcal{F}$ is invariant under G. Then, in the case of the group-theoretic example 1.2 of a coarse fibration (as in G. Mess' theorem), it immediately follows that Y is quasi-isometric to the group G/C, which is therefore, is either Gromov-hyperbolic with topological circle as the ideal boundary, or is virtually nilpotent. This is our proof of G. Mess' theorem. The reader interested only in an alternative proof of [14] can stop reading the paper at this point.

In our situation it is not a priori clear that the base Y is quasi-isometric to a group: the projection of a quasi-action of G from X to Y may not be discrete. Such examples are provided by Solv-manifolds.

In section 6 we restrict somewhat general discussion of coarse fibrations in the preceding sections to the case of coarse fibrations by lines. We introduce the key tool for dealing with the problem of nondiscreteness of the quasi-action of G on the coarse base Y of polynomial growth: the *expansion function* $E(F)$ for the fibers of the coarse fibration $\mathcal{F}$. This function is a coarse analogue of the curvature in the case of Riemannian foliations: it measures the amount of expansion of the metric on a fiber $\bar{F}$ under the nearest-point projection $\bar{F} \to \bar{F}_0$ to a distinguished fiber $\bar{F}_0$. We then assume that the base Y has polynomial growth, in which case we prove:

(a) The function $E : \mathcal{F} \to \mathbb{R}$ is bounded.

(b) Using (a) and amenability of Y we then show that the total space of the coarse fibration is also amenable, i.e. admits a Følner sequence.

(c) We then construct a homomorphism ψ from the group G to the group of quasi-isometries of the fiber $\bar{F}_0$.

(d) 1. In the case when ψ is trivial we use a commutator trick to prove that the fiber $\bar{F}_0$ is Hausdorff-close to a *central infinite cyclic subgroup* $Z \subset G$.

[1] We refer to this as Y being a *quasi-plane*.

2. In the case when ψ is nontrivial, by taking an asymptotic cone of $\bar{F}_0$ we construct a nontrivial homomorphism $G \to \mathbb{Z}$ (or, rather, we get such a homomorphism from a finite-index subgroup in G). Thus, in the context of the universal cover of a closed 3-manifold M, we conclude that M is homotopy-equivalent to a virtually Haken manifold.

In section 7 we prove the main theorem of our paper. In the case when the coarse base Y has polynomial growth, the results of section 6 imply that M is (up to homotopy) virtually Haken with amenable fundamental group, which implies that M is (up to homotopy) either flat, Solv or Nil-manifold.

In the case when the coarse base Y is Gromov-hyperbolic, by considering various classes of cobounded subgroups of $\mathrm{Isom}(\mathbb{H}^2)$: discrete, solvable, dense, we conclude that M is (up to homotopy) either a Seifert manifold with hyperbolic base, or $\pi_1(M)$ is virtually solvable. In the latter case M is homotopy-equivalent to a Solv-manifold.

In section 8 we classify coarse fibrations by lines of Seifert and Solv-manifolds. We show that they are Hausdorff-close to *standard examples* of coarse fibrations, i.e. Seifert fibrations, foliations by affine lines in the case of flat manifolds, and the laminations by lines (described earlier in the introduction) for Solv-manifolds.

In section 9 we prove that each Gromov-hyperbolic group, which admits an invariant coarse fibration by lines, must be virtually cyclic.

Acknowledgments. During the work on this paper the first author was visiting the Max Plank Institute (Bonn), he was also supported by the NSF grants DMS-99-71404 and DMS-02-03045. The second author was supported in part by the NSF Grant DMS-02-24104.

Contents

2. Notation and preliminaries

2.1. **Notation and conventions.** We let $\mathbb{Z}_+ := \{m \in \mathbb{Z} \mid m \geq 0\}$ and $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. We let $d_H(\cdot,\cdot)$ denote the Hausdorff distance between subsets of a metric space; the usual (infimal) distance will be denoted $d(\cdot,\cdot)$. All maps between cell complexes will be continuous unless otherwise specified. Given a map f we let $\mathrm{Im}(f)$ denote the image of f.

We will be using singular (co)homology with $\mathbb{Z}$ coefficients unless we indicate otherwise. For each negative integer k we set $H_k(\cdot) = 0$, $H_c^k(\cdot) = 0$, etc.

Let r_i, R_i be two sequences of positive real numbers. We will use the notation

$$R_i \gtrsim r_i$$

if there exist a pair of constants A, B (independent of i) such that for all but finitely many $i \in \mathbb{N}$ we have:

$$R_i \geq Ar_i + B.$$

We will use the notation $R_i \simeq r_i$ if $R_i \gtrsim r_i$ and $r_i \gtrsim R_i$.

A subset $S \subset Z$ of a metric space is called *δ-dense* if each point $z \in Z$ is within distance $\leq \delta$ from S. A subset in Z which is δ-dense from some $\delta < \infty$, is called a *net* in Z.

A metric space Z has *bounded geometry* if there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each metric ball $B(x,r) \subset Z$, one has:

$$Vol(B(x,r)) := |B(x,r)| \leq \phi(r),$$

where $|S|$ denotes the cardinality of a set S.

In sections 6.4 and 6.6 we will be using *asymptotic cones* and ultralimits of metric spaces. We refer the reader to [12] and [11] for definitions and properties of these constructions.

2.2. **Maps and actions.** Let Y be a subset of a metric space X. A map $f : Y \to X$ has *bounded displacement* if there is $C \in \mathbb{R}$ such that $d(f, i_Y) \leq C$ where $i_Y : Y \to X$ is the inclusion.

A map $f : X \to X'$ between metric spaces is *uniformly proper* if there are constants L, A, and a continuous strictly increasing *distortion function* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to\infty} \eta(t) = \infty$ such that

$$\eta(d(x_1,x_2)) \leq d(f(x_1), f(x_2)) \leq Ld(x_1,x_2) + A \tag{2.1}$$

for all x_1, $x_2 \in X$. A family of maps $\{f_i\}_{i\in\mathcal{J}}$ is *uniformly proper* if one can choose L, A, and η as above so that (2.1) holds for all the maps in the family.

A map $f : X \to X'$ between metric spaces X and X' is an (L, A)-*quasi-isometry* if for all $x_1, x_2 \in X$ we have

$$\frac{1}{L} d(x_1, x_2) - A \leq d(f(x_1), f(x_2)) \leq L d(x_1, x_2) + A$$

and $d(x', \mathrm{Im}(f)) < A$ for all $x' \in X'$. We let $\widehat{\mathrm{QI}}(X, X')$ denote the collection of all quasi-isometries from X to X'. Two quasi-isometries $f_1, f_2 : X \to X'$ are *equivalent* if $d(f_1, f_2) < \infty$; we let $\mathrm{QI}(X, X')$ denote the set of equivalent classes of quasi-isometries, and use $\mathrm{QI}(X)$ (resp. $\widehat{\mathrm{QI}}(X)$) in place of $\mathrm{QI}(X, X)$ (resp. $\widehat{\mathrm{QI}}(X, X)$). Composition of quasi-isometries induces a group structure on $\mathrm{QI}(X)$.

A *quasi-action* of a group G on a metric space X, denoted $G \overset{\rho}{\curvearrowright} X$, is a map $\rho : G \to \widehat{\mathrm{QI}}(X)$ such that for suitable constants L, A,

1. $\rho(g)$ is an (L, A) quasi-isometry for all $g \in G$,
2. $d(\rho(1), \mathrm{id}_X) < A$, and
3. $d(\rho(g_1 g_2), \rho(g_1)\rho(g_2)) < A$ for all g_1, $g_2 \in G$.

We will usually write $g(x)$ rather than $\rho(g)(x)$, suppressing the name of the quasi-action when it is understood. A quasi-action is *discrete* if for all $x \in X$, $R > 0$, the set $\{g \in G \mid d(g(x), x) < R\}$ is finite. A quasi-action $G \overset{\rho}{\curvearrowright} X$ is called *cobounded* if for some (for every) point $x \in X$ the *quasi-orbit* $G \cdot x = \{g(x) : g \in G\}$ is a net in X.

A *degenerating quasi-action* of a finitely-generated group G on a metric space X is a homomorphism $\rho : G \to \mathrm{QI}(X)$. Note that each quasi-action of $G \curvearrowright X$ defines a degenerating quasi-action. The *kernel* of a (degenerating) quasi-action is the kernel of the homomorphism $\rho : G \to \mathrm{QI}(X)$; it consists of the elements of G which act on X with bounded displacement.

Suppose that $G \overset{\phi}{\curvearrowright} Y$ is a quasi-action with the kernel K. Note that a priori there is no uniform bound on the displacement functions $d_g = d(y, \phi(g)y)$ (independent on $g \in K$). For instance, for any quasi-action $G \curvearrowright \mathbb{R}^n$, where G *acts* by translations, the kernel is the entire group G. However, if Y is a Gromov-hyperbolic geodesic metric space whose ideal boundary consists of at least 3 points, then there is a uniform upper bound on the functions $d_g, g \in K$.

Let G, H be groups where H is given a left-invariant metric. A map $\rho : G \to H$ is called a *quasi-homomorphism* if there is a constant C such that for each $g_1, g_2 \in G$ we have:

$$d(\rho(g_1 g_2), \rho(g_1)\rho(g_2)) \leq C.$$

In this paper we will need only quasi-homomorphisms to $\mathbb{R}$.

2.3. **Metric cell complexes.** We will be working with CW complexes endowed with an extra structure. Let X be a CW complex, and $X^{(m)}$ denote its m-skeleton, $m \in \mathbb{Z}_+$. Recall that a subcomplex Y of X is a closed subset which is a union of open cells, such that the boundary of each open cell $\sigma \subset Y$ is contained in Y.

A *control map* for X is a function $p : X \to X^{(0)}$ such that

1. $p\big|_{X^{(0)}} = \mathrm{id}_{X^{(0)}}$,
2. p is constant on open cells in X,
3. $p(x)$ belongs to the smallest subcomplex containing x, for all $x \in X$.

A *morphism* $(X, p) \to (X', p')$ is a skeleton preserving continuous map so that for each $i \in \mathbb{N}$ the diameter of the $p'(\sigma)$ is uniformly bounded (σ are i-cells in $X^{(i)}$).

A *bounded geometry metric cell complex* is a CW complex X equipped with a control map p, whose 1-skeleton $X^{(1)}$ is connected and equipped with a path metric with respect to which all edges have length 1, subject to the condition that every closed cell $\sigma \subset X^{(m)}$ intersects at most $D = D(m)$ closed cells in $X^{(m)}$.

Remark 2.2. Note that for such a complex, the metric space $X^{(0)}$ has bounded geometry in the sense of section 2.1.

To simplify the terminology we will refer to bounded geometry metric cell complexes as simply *metric cell complexes*: the bounded geometry will be assumed by default.

Example 2.3. Suppose that X is a connected (finite-dimensional) simplicial complex so that there exists a number M such that the star of each vertex in X contains at most M simplices. Put a path metric on X so that each simplex is isometric to a regular Euclidean simplex with unit edges. This path metric defines a metric on $X^{(0)}$. Define a control map $p : X \to X^{(0)}$ which sends each simplex in X to one of its vertices. Then the pair (X, p) is a metric cell complex.

We say that a metric cell complex X is Gromov-hyperbolic if its zero-skeleton $X^{(0)}$ is Gromov-hyperbolic.

Let X be a metric cell complex. If $V \subset X^{(0)}$ and $R \in \mathbb{Z}_+$, we denote the closed metric R-neighborhood of V in the 0-skeleton by

$$N_R^{(0)}(V) := \{x \in X^{(0)} \mid d(x, V) \le R\}.$$

Given $m \in \mathbb{Z}_+$, $R \in \mathbb{Z}_+$, and a subcomplex $Y \subset X$, we define the *R-neighborhood of Y in the m-skeleton*, $N_R^{(m)}(Y)$, to be the largest subcomplex of $X^{(m)}$ whose 0-skeleton is $N_R^{(0)}(Y^{(0)})$. Note that

1. $N_0^{(m)}(Y) \supset Y \cap X^{(m)}$,
2. If Y, $Y' \subset X$ are subcomplexes and $R + R' < d(Y^{(0)}, Y'^{(0)})$, then for all $m \in \mathbb{Z}_+$ we have $N_R^{(m)}(Y) \cap N_{R'}^{(m)}(Y') = \emptyset$.

If X is an m-dimensional metric cell complex then we will use the abbreviation $N_R(V) := N_R^{(m)}(V)$.

We will only use the notation $B(x,r)$ to denote closed metric balls in the 1-skeleton $X^{(1)}$, i.e. $B(x,r) := \{y \in X^{(1)} \mid d(x,y) \leq r\}$, where $x \in X^{(1)}$. If Y is a subset of X, we define its diameter to be $\mathrm{diam}(Y) := \mathrm{diam}(p(Y))$. Similarly, we define the distance between two functions $f, f' : S \to X$ to be the quantity $d(p \circ f, p \circ f')$.

A (continuous) map $f : (X,p) \to (X',p')$ is *uniformly proper* if for each $m \in \mathbb{Z}_+$ there are constants $L = L_m$, $A = A_m$, and a *distortion function* $\eta = \eta_m : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, strictly monotone, with $\lim_{t\to\infty} \eta(t) = \infty$, such that for all cells $\sigma_1, \sigma_2 \in X^{(m)}$,

$$\eta(d(p(\sigma_1), p(\sigma_2)) \leq d(p'f(\sigma_1), p'f(\sigma_2)) \leq Ld(p(\sigma_1), p(\sigma_2)) + A. \tag{2.4}$$

A family of maps $\{f_i\}_{i\in\mathcal{J}}$ is *uniformly proper* if one can choose L_m, A_m, and η_m as above so that (2.4) holds for all the maps in the family.

A (continuous) map $f : X \to X'$ between metric cell complexes (X,p) and (X',p') is an (L_m, A_m)*-quasi-isometry* ($m \in \mathbb{Z}_+$) if for all $\sigma_1, \sigma_2 \in X^{(m)}$ we have

$$\frac{1}{L_m} d(p(\sigma_1), p(\sigma_2)) - A_m \leq d(p'f(\sigma_1), p'f(\sigma_2)) \leq L_m d(p(\sigma_1), p(\sigma_2)) + A_m$$

and $d(x', \mathrm{Im}(f^{(m)})) < A_m$ for all $x' \in X'^{(m)}$.

A metric cell complex (X,p) is *uniformly k-connected* if there is a function $\theta = \theta_k : \mathbb{Z}_+ \to \mathbb{Z}_+$ such that the inclusion $N_r^{(k)}(x) \to N_{\theta(r)}^{(k+1)}(x)$ is null-homotopic for all $x \in X^{(0)}$, $r \in \mathbb{Z}_+$. X is *uniformly contractible* if it is uniformly k-connected for all k.

If we are given a collection $\mathfrak{X}$ of bounded geometry cell complexes X, then we will say that $\mathfrak{X}$ has bounded geometry (resp. is uniformly k-connected, uniformly contractible) if each $X \in \mathfrak{X}$ has bounded geometry (resp. is uniformly k-connected, uniformly contractible) where the functions $D(m)$ (resp. θ_k) may be chosen independently of $X \in \mathfrak{X}$.

Remark 2.5. When X is a uniformly contractible cell complex, given two continuous maps f_1, f_2 from a finite dimensional cell complex Z into X where $d(f_1, f_2) < D$, there is a homotopy from f_1 to f_2 whose tracks have diameter $< D'$ where $D' = D'(D, \dim Z)$.

2.4. **Category $\mathcal{C}_B$ of complexes with bounded control.** For the discussion of coarse fibrations it is convenient to introduce a *category $\mathcal{C}_B$ of cell complexes with bounded control over a metric space B*. This category generalizes the category of metric cell complexes.

Fix a metric space B (in all applications B will have bounded geometry). The objects of the category $\mathcal{C}_B$ are pairs (W, c) where W is a CW-complex and $c : W \to B$ is a not necessarily continuous *control* map. We note that B does not have to be equal to $W^{(0)}$.

Example 2.6. Let (W,p) be a metric cell complex. Set $B := W^{(0)}$, $c_B := p$.

A *morphism* between (W, c) and (W', c') is a continuous, skeleton-preserving map $\mu : W \to W'$ which is proper on each skeleton, and has (skeleton-wise) bounded control over B, i.e.

$$d(c' \circ \mu\big|_{W^{(k)}}, c\big|_{W^{(k)}})$$

is finite for each k.

2.5. **Growth and amenability.** A bounded geometry metric space Z has *polynomial growth* if there is a constant $c \in \mathbb{R}_+$ such that for each ball $B(x,r) \subset X$ one has $r^c \gtrsim Vol(B(x,r))$. The optimal constant c is called the *degree of the polynomial growth.* A space Z has *superpolynomial growth* if it does not have polynomial growth for any c.

Similarly, a bounded geometry metric space Z has *exponential growth* if there is a constant $a > 0$ such that for each $x \in Z$ one has:

$$Vol(B(x,R)) \gtrsim e^{aR}.$$

A space has *subexponential growth* if

$$e^{aR} \gtrsim Vol(B(x,R))$$

for all $a > 0$.

A metric space Z is *doubling* if there is a constant $N \in \mathbb{Z}_+$ such that each ball $B(x, 2R) \subset Z$ (where $R \geq 1$) is contained in the union of $\leq N$ balls of radius R.

Lemma 2.7. *(See e.g.* [10]*.) Suppose that Z is a bounded geometry doubling metric space. Then Z has polynomial growth.*

There is another closely related concept, *amenability.* Assume that Z has bounded geometry metric space. For a subset $D \subset Z$ define the *c-frontier* $\partial_c D$ as

$$\partial_c D := \{x \in Z \setminus D : d(x, D) \leq c\}.$$

A sequence $D_j \subset Z$ is called a *c-Følner sequence* if

$$\lim_{j\to\infty} Vol(D_j) = \infty, \text{ and } \lim_{j\to\infty} \frac{Vol(\partial_c D_j)}{Vol(D_j)} = 0.$$

A metric space Z is called *amenable* if it admits a c-Følner sequence for each c. It is clear that amenability is a quasi-isometry invariant property for bounded geometry metric spaces. A finitely-generated group is amenable iff it has an invariant mean. Examples of amenable groups are given by solvable groups. Free nonabelian groups provide examples of nonamenable groups. Moreover, if a group G contains a nonamenable subgroup or maps onto a nonamenable group, then G is nonamenable.

We note that for the metric spaces Z which are vertex sets of connected graphs with unit edges (e.g., finitely generated groups with their word metrics), to test amenability of Z it suffices to consider the 1-frontiers $\partial D := \partial_1 D$ in the definition of Følner sequences.

Lemma 2.8. *If the space Z has subexponential growth then is amenable.*

Proof. Pick a point $x \in Z$ and consider the sequence $D_j := B(x, j) \subset Z$, $j \in \mathbb{N}$. Suppose that the sequence (D_j) contains no c-Følner subsequence. Then there is a constant $C > 0$ such that for each j, $Vol(\partial_c D_j) \geq C\, Vol(D_j)$, i.e.

$$Vol(B(x, j + c)) \geq (C + 1)Vol(B(x, j)).$$

Then $Vol(B(n)) \geq (C + 1)^n Const$ and hence Z has exponential growth. □

3. Coarse fibrations

3.1. **Coarse fibrations.** Let $\bar{\mathcal{F}}$ be a collection of subsets of a metric space Z. Then $\bar{\mathcal{F}}$ defines a *coarse fibration* of Z if there is a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

1. For all $\bar{F}$, $\bar{F}' \in \bar{\mathcal{F}}$,

$$d_H(\bar{F}, \bar{F}') \leq \psi(d(\bar{F}, \bar{F}')), \tag{3.1}$$

where d_H denotes the Hausdorff distance.

2.

$$\bigcup_{\bar{F} \in \bar{\mathcal{F}}} \bar{F}$$

is a net in Z.

We will refer to elements of $\bar{\mathcal{F}}$ as *fibers* of the coarse fibration, and to the function η as an *oscillation function*. In case when Z is 0-skeleton of a metric cell complex X, by abusing notation we will also refer to $\bar{\mathcal{F}}$ as a coarse fibration of X.

Examples:

1. Let $h : Z \to B$ be a Riemannian submersion, where Z is a connected, complete Riemannian manifold. Then $\{h^{-1}(b)\}_{b \in B}$ is a coarse fibration of Z. More generally, if $h : Z \to B$ is a smooth map between complete connected Riemannian manifolds, and there is a constant C such that for all $x \in Z$, $\|Dh(x)\| < C$ and $Dh(x) : T_x Z \to T_{h(x)} B$ has a right inverse $T_{h(x)} B \to T_x Z$ with norm $< C$, then $\{h^{-1}(b)\}_{b \in B}$ is a coarse fibration. To see this, note that the conditions imply that h is Lipschitz, and that there is a smooth distribution transverse to the fibers of h (a "connection"), so that unit speed paths in B can be lifted horizontally to paths in Z with uniformly bounded speed. This leads easily to (3.1).

2. If $G \overset{\rho}{\curvearrowright} Z$ is a cobounded quasi-action on a metric space Z, then the collection of quasi-orbits $\{G \cdot x\}_{x \in Z}$, is a quasi-fibration of Z.

3. Suppose that we are given a finite collection of geodesic metric spaces $F_i, i \in \mathcal{J}$, and for each pair $i, j \in \mathcal{J}$, we are given an (L, A)-quasi-isometry $f_{ij} : F_i \to F_j$ satisfying

$$d(f_{ij} \circ f_{jk}, f_{ik}) \leq C.$$

Let Γ be a (connected) metric graph with edges of the unit length and suppose that $\phi : \Gamma^{(0)} \to \mathcal{J}$ is a map. Let Z denote the disjoint union of the metric spaces $\bar{F}_v = (F_{\phi(v)}, v)$, $v \in \Gamma^{(0)}$. We metrize Z as follows: For every edge $[v, w]$ in Γ we connect each $x \in \bar{F}_v$ to $f_{\phi(v)\phi(w)}(x)$ by an edge of the unit length. This, together with the geodesic metrics on F_i's, determine a path metric on Z. The collection of subsets $\bar{F}_v$, $v \in \Gamma^{(0)}$, determine a coarse fibration of Z.

If $\bar{\mathcal{F}}$ is a coarse fibration of Z, we (pseudo)metrize $\bar{\mathcal{F}}$ using Hausdorff distance, and refer to the resulting metric space B as the *base* of the coarse fibration. When Z is a length space, $(\bar{\mathcal{F}}, d_H)$ is a quasi-isometric to a length space. To see this, observe that given two fibers $\bar{F}$, $\bar{F}' \in \bar{\mathcal{F}}$, and a 1-Lipschitz path $\gamma : [0, l] \to Z$ of length $l \in \mathbb{Z}$,

$$d(\bar{F}, \bar{F}') - 1 < l < d(\bar{F}, \bar{F}') + 1,$$

joining $\bar{F}$ and $\bar{F}'$, we can choose fibers $\bar{F}_i$ at uniform distance from $\gamma(i)$, and then we get

$$\sum_{i=1}^{l} d_H(\bar{F}_{i-1}, \bar{F}_i) < Cl < C d_H(\bar{F}, \bar{F}')$$

where C is independent of $\bar{F}$, $\bar{F}'$.

If $\bar{F}$, $\bar{F}' \in \bar{\mathcal{F}}$ are two fibers (given metrics induced from Z) and $\phi : \bar{F} \to Z$ is a bounded displacement map with $\operatorname{Im} \phi \subset \bar{F}'$ (e.g. any nearest point projection $\bar{F} \to \bar{F}'$), then ϕ is an (L, A)-quasi-isometry between $\bar{F}$ and $\bar{F}'$, where L and A depend only on the displacement of ϕ and the distortion function for $\bar{\mathcal{F}}$.

We say that two coarse fibrations $\bar{\mathcal{F}}$ and $\bar{\mathcal{L}}$ are Hausdorff-close if there exists a constant $C < \infty$ such that for each $\bar{F} \in \bar{\mathcal{F}}$ there exists $\bar{L} \in \bar{\mathcal{L}}$ so that $d_H(\bar{F}, \bar{L}) \leq C$, and vice-versa.

Definition 3.2. If $G \curvearrowright Z$ is a quasi-action on Z, then the coarse fibration $\bar{\mathcal{F}}$ is *G-invariant* if there is a constant C such that for all $g \in G$, $\bar{F} \in \bar{\mathcal{F}}$, there is an $\bar{F}' \in \bar{\mathcal{F}}$ such that $d_H(g(\bar{F}), \bar{F}') < C$.

Note that this definition is slightly more general than definition 1.1 in the introduction.

For each coarse fibration $\bar{\mathcal{F}}$ which is invariant under a quasi-action $G \curvearrowright Z$, we obtain a natural quasi-action $G \curvearrowright (\bar{\mathcal{F}}, d_H)$ of G on the base: Given $g \in G$ and a point $x \in B$ represented by a fiber $\bar{F} \in \bar{\mathcal{F}}$, we let $g \cdot x$ to be a point in B represented by the fiber $\bar{F}' \in \bar{\mathcal{F}}$ such that $d_H(g(\bar{F}), \bar{F}') < C$. Thus we have:

Lemma 3.3. *Suppose that $G \curvearrowright Z$ is a quasi-action such that $\bar{\mathcal{F}}$ is invariant under G. Then the quasi-action $G \curvearrowright Z$ projects to a quasi-action $G \curvearrowright B$; the latter is cobounded provided that the quasi-action $G \curvearrowright Z$ is cobounded.*

We also get a *degenerating quasi-action* $G \overset{\psi}{\curvearrowright} \bar{F}_0$ on a fiber $\bar{F}_0 \in \bar{\mathcal{F}}$ as follows: Pick a nearest point projection $r_{\bar{F}_0} : Z \to \bar{F}_0$. Then we define a degenerating quasi-action ψ by composing $g\big|_{\bar{F}_0} : \bar{F}_0 \to Z$ with $r_{\bar{F}_0}$. The quasi-isometry constant of $\psi(g)$ depends only on $d_H(g(\bar{F}_0), \bar{F}_0)$. In particular, we obtain a homomorphism $G \to \mathrm{QI}(\bar{F}_0)$.

Suppose $\mathcal{M}$ is a collection of metric spaces. A coarse fibration $\bar{\mathcal{F}} = \{\bar{F}_j\}_{j \in J}$ of a metric space Z is *by elements of* $\mathcal{M}$ if one can choose, for each j, an element $M_j \in \mathcal{M}$ and a map $f_j : M_j \to Z$ such that $\mathrm{Im}(f_j) = \bar{F}_j$ for all $j \in J$ and the family $\{f_j\}$ is uniformly proper.

If $\mathcal{A}$ is a collection of metric cell complexes, then a coarse fibration $\bar{F}$ of Z is *by elements of* $\mathcal{A}$ if $\bar{F}$ is a coarse fibration by elements of $\{A^{(0)} \mid A \in \mathcal{A}\}$.

3.2. **Parameterizing coarse fibrations.** In the sequel it will be convenient to specify parameterizations of the fibers of a coarse fibration. Suppose $\bar{\mathcal{F}} = \{\bar{F}_j\}_{j \in \mathcal{J}}$ is a coarse fibration of the 0-skeleton $X^{(0)}$ of a metric cell complex X. A *parametrization* of $\bar{\mathcal{F}}$ is a collection of pairs $\{(F_j, f_j)\}_{j \in \mathcal{J}}$ where

1. For each $j \in \mathcal{J}$, F_j is a bounded geometry cell complex, and the family $\{F_j\}_{j \in \mathcal{J}}$ has bounded geometry,
2. For each $j \in \mathcal{J}$, f_j is a morphism from F_j to X, where $f_j^{(0)} : F_j^{(0)} \to X^{(0)}$ satisfies $\mathrm{Im}(f_j^{(0)}) = \bar{F}_j$, and
3. The family of morphisms $\{f_j\}_{j \in \mathcal{J}}$ is uniformly proper.

Definition 3.4. We will refer to a coarse fibration $\bar{\mathcal{F}}$ together with a choice of parametrization $\{(F_j, f_j)\}_{j \in \mathcal{J}}$ as a *parameterized coarse fibration.* We denote the parameterized coarse fibration by $\mathcal{F}$. We will often refer to the F_j's as *fibers* of $\mathcal{F}$.

The next lemma shows that in some cases one can construct a parametrization for a given coarse fibration.

Lemma 3.5. *Suppose X is a metric cell complex, and $\bar{\mathcal{F}}$ is a coarse fibration of $X^{(0)}$ by elements of a collection $\mathcal{A}$ of cell complexes with uniformly bounded geometry. If X is uniformly $(k-1)$-connected for all $k \leq \sup\{\dim(A) \mid A \in \mathcal{A}\}$, then $\bar{\mathcal{F}}$ admits a parametrization $\mathcal{F} = \{(F_j, f_j)\}_{j \in \mathcal{J}}$ where $F_j \in \mathcal{A}$ for all $j \in \mathcal{J}$. If in addition $\mathcal{A}$ is uniformly $(k-1)$-connected, then for each $(F, f) \in \mathcal{F}$, there is a (continuous) map $v_F^{(k)} : X^{(k)} \to F$ such that the composition $N_R^{(k)}(\bar{F}) \overset{v_F^{(k)}}{\to} F^{(k)} \overset{f}{\to} X^{(k)}$ is at distance $< D = D(k, R)$ from the inclusion map. In particular, $v_F^{(1)}$ is always defined since metric cell complexes are (linearly) uniformly connected.*

Proof. One constructs the maps f_j and $v_F^{(k)}$ by induction on skeleta using the assumption of uniform $(k-1)$-connectedness and the nearest-point projection $f \circ v_F^{(0)} : X^{(0)} \to \bar{F}$ to begin the induction. $\square$

The following lemma is elementary and we leave it to the reader:

Lemma 3.6. *Suppose that $\mathcal{F}$ is a parameterized coarse fibration of X. Assume in addition that there is a uniform bound on the dimension of the fibers F_i and that X is uniformly contractible. Then there exists a parameterized coarse fibration $\mathcal{L}$ of X such that:*

(a) Distinct fibers of $\bar{\mathcal{L}}$ are disjoint.

(b) $\bar{\mathcal{L}}$ is Hausdorff-close to $\bar{\mathcal{F}}$.

(c) The union of fibers in $\bar{\mathcal{L}}$ equals $X^{(0)}$.

The following lemma is immediate from the definitions.

Lemma 3.7. *Let Z_0 be a metric space.*

1. If Z and Z' are length spaces, $f : Z \to Z_0$, $f' : Z' \to Z_0$ are uniformly proper maps, and $Z_0 \subset N_r(\mathrm{Im}(f)) \cap N_r(\mathrm{Im}(f'))$ for some $r \in \mathbb{R}_+$, then there is an (L, A) quasi-isometry $\phi : Z \to Z'$ so that $d(f' \circ \phi, f) < D$ where L, A, D depend only on r and the distortion data of f and f'.

2. If X and X' are uniformly contractible metric cell complexes, and $f : X^{(0)} \to Z_0$, $f' : X'^{(0)} \to Z_0$ are surjective uniformly proper maps, then there is an (L, A) quasi-isometry $\phi : X \to X'$ so that $d(f' \circ \phi^{(0)}, f) < D$ where L, A, D depend only on the distortion data of f and f'.

Proposition 3.8. *Let $\mathcal{F}$ be a parameterized coarse fibration of the zero skeleton $Z := X^{(0)}$ of a metric cell complex.*

1. If (F_1, f_1), $(F_2, f_2) \in \mathcal{F}$ and $\psi : \bar{F}_1 \to X^{(0)}$ is a bounded displacement map with image contained in $\bar{F}_2$, then there is a quasi-isometry $\phi : F_1^{(0)} \to F_2^{(0)}$ such that

$$d(\psi \circ f_1, f_2 \circ \phi) < D,$$

where D and the quasi-isometry constants depend only on the displacement of ψ.

2. Suppose $G \curvearrowright Z$ is a quasi-action preserving $\bar{\mathcal{F}}$. Then each $g \in G$ induces a quasi-isometry $(\bar{\mathcal{F}}, d_H) \xrightarrow{\bar{g}} (\bar{\mathcal{F}}, d_H)$. We also obtain, for each $(F, f) \in \mathcal{F}$, a degenerating quasi-action $G \overset{\psi}{\curvearrowright} F^{(0)}$ by sending $g \in G$ to the composition

$$F^{(0)} \xrightarrow{f} X^{(0)} \xrightarrow{g} X^{(0)} \xrightarrow{v_F^{(0)}} F^{(0)}.$$

Here the quasi-isometry constant of $\psi(g)$ depends only on $d_H(g(\bar{F}), \bar{F})$.

Proof. Let ψ be as in 1. Then $\mathrm{Im}(\psi \circ f_1^{(0)})$ is D_1-dense in $\mathrm{Im}(f_2^{(0)})$, where D_1 depends only on the displacement δ of ψ. We can perturb $f_2^{(0)}$ to a map h so that $\mathrm{Im}(h) = \mathrm{Im}(\psi \circ f_1^{(0)})$, and h is uniformly proper with distortion controlled by D_1 (and hence by δ). Now part 2 of Lemma 3.7 applies.

Part 2 follows from Lemma 3.7 and the corresponding assertions (see Lemma 3.3 and the discussion that follows it) for unparametrized coarse fibrations. □

As a corollary of the above lemma, we get a topological action of G on the space of ends of the fiber F_0.

Definition 3.9. In case when F_0 is 2-ended, we say that the degenerating quasi-action ψ *preserves orientation on* F_0 if G acts trivially on the space of ends of F_0.

Note that degenerating quasi-action of G preserves orientation on one fiber F_0 iff it preserves orientation on any other fiber F_i.

4. Coarse fibrations and product structure

In this section we will show that coarse fibration behave homotopy theoretically like products when the total space and the fibers are (the zero skeleta of) uniformly contractible metric cell complexes.

Let (X, p) be a uniformly contractible metric cell complex, and suppose $\bar{\mathcal{F}}$ is a coarse fibration of $X^{(0)}$ by a uniformly contractible, bounded geometry family $\mathcal{A}$ of metric cell complexes. Using Lemma 3.5, $\bar{\mathcal{F}}$ can be given a parametrization $\mathcal{F} = \{(F_j, f_j)\}_{j \in \mathcal{J}}$, where $F_j \in \mathcal{A}$.

Let $Y^{(0)}$ be a net in $(\bar{\mathcal{F}}, d_H)$ with the property that all R-balls in $Y^{(0)}$ have cardinality $< C = C(R)$. Let $c_X : X \to Y^{(0)}$ be a (discontinuous) projection map given by letting $c_X(x) \in Y^{(0)}$ be a fiber nearest to $p(x)$. The pair (X, c_X) defines an object in the category $\mathcal{C}_{Y^{(0)}}$ (see section 2 for the definition of bounded control categories). Pick $(F, f) \in \mathcal{F}$. If V is any CW-complex based on $Y^{(0)}$ with control map $c_V : V \to Y^{(0)}$, we get another object in $\mathcal{C}_{Y^{(0)}}$ by taking the pair $(V \times F, c_{V\times F})$ where $c_{V\times F}$ is the composition $V \times F \to V \xrightarrow{c_V} Y^{(0)}$.

Our main goal in this section is:

Proposition 4.1. *There is a bounded geometry, uniformly contractible CW complex Y based on $Y^{(0)}$ with control map $c_Y : Y \to Y^{(0)}$, and a morphism $\phi : (Y \times F, c_{Y\times F}) \to (X, c_X)$ which is a homotopy equivalence in the category $\mathcal{C}_{Y^{(0)}}$.*

Definition 4.2. We will refer to Y as a *coarse base* of the coarse fibration $\mathcal{F}$.

The proof of the Proposition 4.1 occupies the rest of this section. The proof is by induction on the skeleta of Y. We start by noting that since $Y^{(0)}$ is a quasi-isometric to a length space (see section 3.1), there is a metric graph $Y^{(1)}$ (based on $Y^{(0)}$) so that the inclusion map $(Y^{(0)}, d_H) \to Y^{(1)}$ is a quasi-isometry. The graph $Y^{(1)}$ has bounded geometry because each R-ball in $(Y^{(0)}, d_H)$ has cardinality $\leq C(R)$. To continue with the induction we will need several auxiliary lemmas.

Lemma 4.3. *Pick $k \in \mathbb{Z}_+$. For every R there is an $R' = R'(R)$, and a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $\bar{F} \in \mathcal{F}$, any map $\phi : S^k \to N_R^{(k)}(\bar{F})$ with $\operatorname{diam}(\operatorname{Im}\phi) < D$, admits a null-homotopy $\phi \cong const$ with image in $N_{R'}^{(k+1)}(\bar{F}) \cap N_{\alpha(D)}^{(k+1)}(\operatorname{Im}\phi)$.*

Proof. Let R be given, and let $\phi : S^k \to N_R^{(k)}(\bar{F})$ be a continuous map so that $\operatorname{diam}(\operatorname{Im}\phi) < D$. The image of the composition $v_F \circ \phi : S^k \to F$ has diameter $< D_1 = D_1(D, R)$. By the uniform contractibility of F, there is a null-homotopy $H : S^k \times I \to F$ of $v_F \circ \phi$ whose image $\operatorname{Im}(H)$ is contained in the $D_2 = D_2(D, R)$ neighborhood of $\operatorname{Im}(v_F \circ \phi)$ in the $(k+1)$-skeleton of X. By Remark 2.5, the map ϕ is homotopic to $f \circ v_F \circ \phi$ by a homotopy whose tracks have diameter $< D_3 = D_3(D, R)$. Combining this homotopy with the homotopy $f \circ H$, we obtain the desired null-homotopy. □

Let V be a metric cell complex based on $Y^{(0)}$ with a control map $c_V : V \to Y^{(0)}$. Pick $(F, f) \in \mathcal{F}$, and let $c_{V \times F}$ be as indicated above, so that $(V \times F, c_{V \times F})$ is an object in the category $\mathcal{C}_{Y^{(0)}}$.

Lemma 4.4. *There is a morphism $\Phi : (V \times F, c_{V \times F}) \to (X, c_X)$ in the category $\mathcal{C}_{Y^{(0)}}$ with the following property. For every R-ball $B \subset Y^{(0)}$, the restriction of $v_F \circ \Phi$ to $\left(c_V^{-1}(B) \cap V^{(k)}\right) \times F^{(l)}$ is at distance at most $D = D(k, l, R, d(B, c_X(\bar{F}))$ from the projection $\left(c_V^{-1}(B) \cap V^{(k)}\right) \times F^{(l)} \to F$.*

Proof. We construct Φ by induction on the skeleta of V. We define $\Phi^{(0)} : V^{(0)} \times F \to X$ so that for each $y = (F', f') \in V^{(0)} = Y^{(0)}$, the restriction of $\Phi^{(0)}$ to $y \times F$ agrees with the composition $F \xrightarrow{f} X \xrightarrow{v_{F'}} F' \xrightarrow{f'} X$.

We continue inductively. To proceed from $V^{(k)}$ to $V^{(k+1)}$, for each cell $\Delta^{(k+1)}$ in $V^{(k+1)}$ we construct a map $\Delta^{(k+1)} \times F \to X$ by extending the given map $(\partial \Delta^{(k+1)}) \times F \to X$. This extension is done by induction on skeleta in F via Lemma 4.3. □

Lemma 4.5. *For every $n \in \mathbb{Z}_+$, there is a bounded geometry cell complex $Y^{(n)}$ based on $Y^{(0)}$ with control map $c : Y^{(n)} \to Y^{(0)}$ so that $Y^{(n)}$ is uniformly $(n-1)$-connected.*

Proof. Assume inductively that a uniformly $(k-1)$-connected bounded geometry cell complex $Y^{(k)}$ has been constructed.

We note that given $D \in \mathbb{R}_+$, there exists a collection of "singular k-spheres" $h_j : S^k \to Y^{(k)}$ so that:

1. The diameters $\operatorname{diam}(\operatorname{Im}(h_j))$ are uniformly bounded.
2. The collection of sets $\operatorname{Im}(h_j), j \in \mathbb{N}$, is uniformly locally finite.
3. Any map $\phi : S^k \to Y^{(k)}$, whose image has diameter $< D$, is null-homotopic in

$$W^{(k+1)}(D) := Y^{(k)} \cup (\cup_{h_j} e_j).$$

Here e_j's are $k+1$–cells which we attach to $Y^{(k)}$ via the maps h_j.

We give $W^{(k+1)}(D)$ structure of a metric cell complex by extending the control map $p: Y^{(k)} \to Y^{(0)}$ to the $k+1$-cells e_j: send each point of $e_j \setminus \partial e_j$ to a point $y \in \mathrm{Im}(h_j) \cap Y^{(0)}$.

Remark 4.6. Arguing inductively, on the skeleta of the sphere S^k, using the above properties of $W^{(k+1)}$ one proves the following. Given D there is a $D_1 = D_1(D)$ such that any two maps $\phi, \phi' : S^k \to Y^{(k)}$ with $d(\phi, \phi') < D$ become homotopic in $W^{(k+1)}(D_1)$, where the tracks of the homotopy have diameter $< D_2 = D_2(D)$.

Using the uniform $(k-1)$-connectedness of $Y^{(k)}$ and induction on skeleta, we may construct a morphism $X^{(k)} \overset{\pi^{(k)}}{\to} Y^{(k)}$ (in the category $\mathcal{C}_{Y^{(0)}}$) which extends $c_X\big|_{X^{(0)}}$. Since X is a bounded geometry cell complex, the composition $X^{(k)} \overset{\pi^{(k)}}{\to} Y^{(k)} \to W^{(k+1)}(D)$ extends to a morphism $X^{(k+1)} \overset{\pi^{(k+1)}}{\longrightarrow} W^{(k+1)}(D)$ when $D > R_1 = R_1(k)$.

Pick $(F, f) \in \mathcal{F}$, and point $t \in F^{(0)}$. Let $\Phi : Y^{(k)} \times F \to X$ be the morphism provided by Lemma 4.4 when applied with $V = Y^{(k)}$. Let $\Psi : Y^{(k)} \to X^{(k)}$ be the composition of the inclusion $Y^{(k)} \to Y^{(k)} \times \{t\}$ with Φ; note that Ψ is a morphism in the category $\mathcal{C}_{Y^{(0)}}$.

Consider a map $\phi : S^k \to Y^{(k)}$, and the composition $\hat{\phi} := \Psi \circ \phi : S^k \to X^{(k)}$. By Lemma 4.3, the map $\hat{\phi}$ is null-homotopic via a null-homotopy $H : S^k \times I \to X^{(k+1)}$ such that $\pi^{(k+1)} \circ H$ has image diameter $< \theta(\mathrm{diam}(\mathrm{Im}(\phi)))$ where $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ depends only on k. Since $d(\pi^{(k)} \circ \hat{\phi}, \phi) < R_2 = R_2(k)$, we conclude by Remark 4.6 that ϕ is homotopic to $\pi^{(k)} \circ \hat{\phi}$ within $W^{(k+1)}(D)$ whenever $D > D_1(R_2)$. Therefore ϕ is null-homotopic in $W^{(k+1)}(D)$ whenever $D > R_3(k)$; furthermore, the diameter of the homotopy is at most $\theta_1(\mathrm{diam}(\mathrm{Im}(\phi)))$ where θ_1 depends only on k. □

By the preceding lemma, we may construct a uniformly contractible metric cell complex Y based on $Y^{(0)}$ with control map $c : Y \to Y^{(0)}$. Pick $(F, f) \in \mathcal{F}$. We apply Lemma 4.4 to construct a morphism $\Phi : Y \times F \to X$.

Lemma 4.7. *We let $\Phi : Y \times F \to X$ be as above. Then Φ admits a homotopy inverse $\tilde{\Phi}$ in the category $\mathcal{C}_{Y^{(0)}}$.*

Proof. Since the metric cell complex Y is uniformly contractible, we can construct a morphism $h : (X, c_X) \to (Y, c_Y)$ by induction on skeleta starting with the map $c_X : X^{(0)} \to Y^{(0)}$. We then let $\tilde{\Phi} : (X, c_X) \to (Y \times F, c_{Y \times F})$ be given by $\tilde{\Phi} := (h, v_F)$. We verify that $\Phi, \tilde{\Phi}$ are homotopy-inverse to each other in the category $\mathcal{C}_{Y^{(0)}}$ using uniform contractibility of Y, F and Lemma 4.3.

□

Lemma 4.7 completes the proof of Proposition 4.1.

5. Coarse fibrations and coarse Poincaré duality

Let X, $\mathcal{F}$, Φ, $\tilde{\Phi}$, and Y be as in the previous section. Assume in addition that X is homeomorphic to an n-manifold, and that every $(F, f) \in \mathcal{F}$ satisfies

$$H_c^*(F^{(i+1)}) \cong H_c^*(\mathbb{R}^i).$$

We will show that under these assumptions the complex Y behaves coarsely like a manifold of dimension $n - i$.

We recall that X satisfies the Poincaré duality:

Theorem 5.1. *For each closed subset $W \subset X$ and $k \in \mathbb{Z}$ there is an isomorphism*

$$P_{W,k} : \check{H}_c^k(W) \to H_{n-k}(X, X \setminus W)$$

which is local in the following sense: $\mathrm{Supp}(P_{W,k}(\tau)) \subset N_{D_X}(\mathrm{Supp}(\tau))$ *for each $\tau \in Z_c^k(W)$. The constant D_X does not depend on W and τ. The family $\{P_{W,k}\}$ is compatible with homomorphisms induced by inclusions.*

Here and in what follows $\check{H}_c^*(\cdot)$ denotes the Čech cohomology.

Recall that $\mathcal{F}$ is a parameterized coarse fibration on n-dimensional uniformly contractible manifold X so that every fiber $(F, f) \in \mathcal{F}$ satisfies

$$H_c^*(F^{(i+1)}) \cong H_c^*(\mathbb{R}^i),$$

and Y is a *coarse base* of the coarse fibration $\mathcal{F}$. Set $j := n - i$.

The following is an analog of the fact that if B is the base of a fibration whose total space and fibers are aspherical manifolds, then B is an aspherical manifold.

Theorem 5.2. *(Y satisfies j-dimensional coarse Poincaré duality.) Pick $k \in \mathbb{Z}$, and set $m := 1 + \max(k, j - k)$. For every subcomplex $K \subset Y^{(m)}$, there exists a homomorphism*

$$P_K : H_c^k(K) \to H_{j-k}(Y^{(m)}, Y^{(m)} \setminus K)$$

so that the collection $\{P_K\}$ has the following properties for a suitable $D \in \mathbb{R}$:

1. $\{P_K\}$ is compatible with inclusions $K_1 \subset K_2 \subset Y^{(m)}$.

2. The kernel of $P_{N_D^{(m)}(K)}$ is contained in the kernel of the restriction $H_c^k(N_D^{(m)}(K)) \to H_c^k(K)$ (i.e., P_K is an approximate monomorphism).

3. The image of the inclusion

$$H_{j-k}(Y^{(m)}, Y^{(m)} \setminus N_D^{(m)}(K)) \to H_{j-k}(Y^{(m)}, Y^{(m)} \setminus K)$$

is contained in the image of P_K (i.e., P_K is an approximate epimorphism).

4. P_K is local in the following sense: if $[\sigma] \in H_c^k(K)$ is represented by a cocycle $\sigma \in Z_c^k(K)$, then $P_K(\sigma)$ can be represented by a relative cycle τ supported in $N_D^{(m)}(\mathrm{Supp}(\sigma))$.

Proof. We construct the maps P_K as follows. Let $[\omega]$ be a generator of $H^i_c(F^{(i+1)})$. First, send each $[\sigma] \in H^k_c(K)$ to the tensor product $[\sigma] \otimes [\omega] \in H^{k+i}_c((K \times F)^{(i+1)})$. Then pull back $[\sigma]\otimes[\omega]$ via $(\tilde{\Phi}^{(m+i)})^*$ to obtain $(\tilde{\Phi}^{(m+i)})^*([\sigma]\otimes[\omega]) \in H^{k+i}_c(\hat{K})$ where $\hat{K} := (\tilde{\Phi}^{(m+i)})^{-1}(K\times F) \subset X^{(m+i)}$ and $\tilde{\Phi}$ is the homotopy-equivalence $\tilde{\Phi} : X \to Y \times F$ constructed in Lemma 4.7 of the previous section. Next, apply $P_{\hat{K},k+i}$ given by Theorem 5.1:

$$[\sigma] \otimes [\omega] \mapsto (\tilde{\Phi}^{(m+i)})^*([\sigma] \otimes [\omega]) \mapsto P_{\hat{K},k+i}((\tilde{\Phi}^{(m+i)})^*([\sigma] \otimes [\omega])) = [\eta].$$

The result is a homology class

$$[\eta] \in H_{n-(k+i)}(X^{(m+i)}, X^{(m+i)} \setminus \hat{K}) = H_{j-k}(X^{(m+i)}, X^{(m+i)} \setminus \hat{K}).$$

Then send $[\eta]$ to $H_{j-k}((Y \times F)^{(m+i)}, (Y \times F)^{(m+i)} \setminus (K \times F)^{(m+i)})$ via $(\tilde{\Phi}^{(m+i)})_*$. Next we apply the isomorphism

$$H_{j-k}(Y^{(m)} \times F, (Y^{(m)} \setminus K) \times F) \simeq H_{j-k}(Y^{(m)}, Y^{(m)} \setminus K).$$

induced by the projection $Y \times F \to Y$. By composing with this isomorphism we get

$$[\eta] \mapsto (\tilde{\Phi}^{(m+i)})_*([\eta]) \mapsto [\tau] \in H_{j-k}(Y^{(m)}, Y^{(m)} \setminus K).$$

As the result we finally get the homomorphism

$$P_K : [\sigma] \mapsto [\eta] \mapsto [\tau].$$

It remains to verify that the P_K's so defined satisfy conditions 1-4 above. Property 1 follows immediately from the fact that each of the factors in the composition defining P_K is compatible with inclusions. Property 4 follows readily from the locality of $P_{\hat{K},k+i}$ and the fact that $\tilde{\Phi}^{(m+i)}$ has bounded control over $Y^{(0)}$. We will only prove property 2, as the proof of 3 is similar. The map $P_K : [\sigma] \mapsto [\tau]$ is a composition of homomorphisms all but two of which are clearly isomorphisms, the exceptions being the pull-back and push-forward via $\tilde{\Phi}$. Since the map $\tilde{\Phi}$ preserves skeleta and X is an n-dimensional manifold, it suffices to restrict to

$$\Phi^{(n)} : Y^{(n)} \times F \to X,\ \tilde{\Phi} : X \to Y^{(n)} \times F\ .$$

We recall that (since Φ and $\tilde{\Phi}$ are homotopy inverses in $\mathcal{C}_{Y^{(0)}}$)

$$\tilde{\Phi} \circ \Phi^{(n)} : Y^{(n)} \times F \to Y^{(n)} \times F$$

is homotopic to $\mathrm{id}_{Y^{(n)}\times F}$ via a homotopy $H : Y^{(n)} \times F \times I \to Y^{(n+1)} \times F$ in the category $\mathcal{C}_{Y^{(0)}}$; projections to $Y^{(0)}$ of the tracks of this homotopy have diameter $\leq D_Y$ for some D_Y. Similarly, (by enlarging D_Y if necessary) $\Phi^{(n)} \circ \tilde{\Phi} : X \to X$ is homotopic to id_X via homotopy $\tilde{H} : X \times I \to X$ in the category $\mathcal{C}_{Y^{(0)}}$; projections to $Y^{(0)}$ of the tracks of this homotopy have diameter $\leq D_Y$. Finally, we assume that D_Y is chosen in such a way that $d(c_X \circ \Phi, c_X) \leq D_Y$ and $d(c_{Y\times F} \circ \tilde{\Phi}, c_{Y\times F}) \leq D_Y$.

Set $D := D_X + 4D_Y$: here D_X is the constant that appears in Theorem 5.1.

Suppose that $R \geq 3D_Y$. If $[\eta] \in H_{j-k}(X, X \setminus \widehat{N_R(K)})$ and $[\eta] \mapsto \tilde{\Phi}_*([\eta]) = 0$ then $[\eta]$ maps to zero under the inclusion induced map

$$H_{j-k}(X, X \setminus \tilde{\Phi}^{-1}(N_R^{(m)}(K) \times F)) \to H_{j-k}(X, X \setminus \tilde{\Phi}^{-1}(N_{R-3D_Y}^{(m)}(K) \times F)).$$

(This follows by noticing that $\Phi_* \circ \tilde{\Phi}_*(\eta) - \eta = d(\theta)$ where the projection of the support of θ to $Y^{(0)}$ has diameter $\leq D_Y$.)

The same argument applies to the map

$$[\sigma] \otimes [\omega] \mapsto \tilde{\Phi}^*[\sigma] \otimes [\omega],$$

where $[\sigma] \otimes [\omega] \in H_c^{k+i}(N_R^{(m)}(K) \times F^{(i+1)})$ for some $R \geq D_Y$. If $[\sigma] \otimes [\omega]$ maps to zero then (again using the homotopy H) we conclude that $[\sigma] \otimes [\omega]$ restricts to zero in $H_c^{i+k}(N_{R-D_Y}^{(m)}(K) \times F^{(i+1)})$.

Combining these two statements and locality of the usual Poincaré duality map, it follows that if $[\sigma] \otimes [\omega] \in H_c^{j+k}(N_D^{(m)}(K) \times F)$ belongs to the kernel of the composition $\Phi_* \circ P \circ \tilde{\Phi}^*$, then $[\sigma] \otimes [\omega]$ belongs to the kernel of the inclusion

$$H_c^{i+k}(N_D^{(m)}(K) \times F) \to H_c^{i+k}(K \times F). \quad \square$$

Corollary 5.3. *For every k we have $H_c^k(Y^{(k+1)}) \cong H_c^k(\mathbb{R}^j)$. If $j \geq 2$ then Y is 1-ended in the sense that $Y^{(1)}$ is 1-ended.*

Corollary 5.4. *If in Theorem 5.2, $n = 3$ and $i = 1$ then the coarse base Y is a quasi-plane, i.e. it is simply-connected and satisfies coarse 2-dimensional Poincaré duality.*

6. Coarse fibrations by lines

In this section $G \curvearrowright Z$ will be a cobounded quasi-action on the 0-skeleton Z of a metric cell complex X, which preserves a parameterized coarse fibration $\mathcal{F} = \{(F_i, f_i)\}_{i \in \mathcal{J}}$ of Z (see section 2). We will assume that the above coarse fibration is *by lines*, i.e. each F_i is isometric to the real line $\mathbb{R}$ with the usual metric and the cell complex structure is given by the triangulation of $\mathbb{R}$ with vertices at the integer points. Let Y be a metric cell complex which is a *coarse base* of the fibration $\mathcal{F}$ (see section 3 for the definition).

For convenience, we will further assume that Z is the disjoint union of the fibers $\bar{F}_i$, $i \in \mathcal{J}$ and that for each $g \in G$, $i \in \mathcal{J}$, there is an $i' \in \mathcal{J}$ such that $g(\bar{F}_i) \subset \bar{F}_{i'}$. We will sometimes denote $F_{i'}$ by $g_* F_i$.

We recall (Lemma 3.8) that for each fiber $(F, f) \in \mathcal{F}$, we have an induced degenerating quasi-action $G \overset{\psi}{\curvearrowright} F$. We denote the image of an element $x \in F$ under $g \in G$ by $\psi_{F_1}(g)$ or $\psi(g)(x)$.

6.1. **The fiber expansion function.** One interpretation of the curvature of a curve sitting in a Riemannian manifold is that it determines the rate of change of length of the curve under normal variation. There is a coarse analog of curvature for the fibers of our parameterized coarse fibration $\mathcal{F}$ on a metric cell complex X. Consider a pair of fibers $\bar{F}_1, \bar{F}_2$. Given an interval $I \subset F_1$, we can measure the length of its image in F_2 under the mapping $v_{F_2} \circ f_1$ where $v_{F_2} : Z \to F_2$ has the property that the composition $f_2 \circ v_{F_2}$ is a nearest-point projection $Z \to \bar{F}_2$, see Lemma 3.5. By taking into account the action of the group, we find that the stretch factor of I under the mapping $v_{F_2} \circ f_1$ is independent of the choice of the interval up to a uniform multiplicative error, provided that I is sufficiently long.

Remark 6.1. This basic construction works more generally for coarse fibrations with uniformly doubling fibers. However instead of the "lengthwise" distortion discussed in this paper one would consider the distortion of volume for domains in the fibers which are quasi-balls with small isopetrimetric ratio. The analogs of Lemma 6.6, Definition 6.11, Corollaries 6.12, 6.13 hold in this context.

We begin by observing that there are many group elements which "almost preserve" any given pair of fibers:

Lemma 6.2. *Let F_1, F_2 be fibers, let $R := d(\bar{F}_1, \bar{F}_2)$, $R \geq 1$. There is a positive constant D (depending on the geometry of the quasi-action $G \curvearrowright Z$ and the coarse fibration $\mathcal{F}$) with the following property. Pick $x \in F_1$, let*

$$\mathcal{G} = \{g \in G \mid d(g(\bar{F}_i), \bar{F}_i) < D \text{ for } i = 1, 2\},$$

and set $\mathcal{G}(x) := \{\psi g(x) \mid g \in \mathcal{G}\} \subset F_1$. Then

1. $\mathcal{G}(x)$ is a net in F_1.

2. If moreover the quasi-action $G \curvearrowright Z$ is discrete, then there are constants L, C (depending only on $G \curvearrowright Z$ and $\mathcal{F}$) such that if m is the number of fibers contained in $N_{LR}(\bar{F}_1)$, then the density of $\mathcal{G}(x)$ in F_1 is at least $\frac{C}{m}$, in the sense that if $I \subset F_1$ is an interval, then the quantity

$$\frac{|\mathcal{G}(x) \cap I|}{|I|} \tag{6.3}$$

is at least $\frac{C}{m}$ when I is sufficiently long.

Proof. The quasi-action $G \curvearrowright X$ is cobounded, so there is a D_0 such that for any x, $x' \in X$, there is a $g \in G$ such that $d(g(x), x') < D_0$. It follows that if

$$\mathcal{G}_0 := \{g \in G \mid d(g(\bar{F}_1), \bar{F}_1) < D_0\},$$

then for any $x \in F_1$, the set

$$\mathcal{G}_0(x) := \{\psi(g(x)) \mid g \in \mathcal{G}_0\}$$

is an r_1-net in F_1 where r_1 is independent of the fibers F_1, F_2 and the point $x \in F_1$.

For each $g \in \mathcal{G}_0$, the image of $\bar{F}_2$ under g^{-1} is contained in a fiber $\bar{F}_{i(g)}$ for some $i(g) \in I$, and $\bar{F}_{i(g)}$ lies in $N_{LR}(\bar{F}_1)$, where L depends only on the constants of the quasi-action $G \curvearrowright Z$. Let $\mathcal{J}_0 \subset \mathcal{J}$ denote the collection of $i(g) \in \mathcal{J}$ as g ranges over $\mathcal{G}_0$, and form a set $\Sigma \subset G$ by choosing, for each $i \in \mathcal{J}_0$, a $\sigma \in G$ such that $\sigma^{-1} \in \mathcal{G}_0$ and $\sigma(\bar{F}_2) \subset \bar{F}_i$. The set Σ consists of the elements σ as above. Notice that for each pair $g \in \mathcal{G}_0$, $\sigma \in \Sigma$ such that $\sigma(\bar{F}_2)$ and $g^{-1}(\bar{F}_2)$ are contained in the same fiber, we have

$$d_H(g\sigma(\bar{F}_i), \bar{F}_i) < D \quad \text{for } i = 1, 2,$$

where D is independent of F_1, F_2, and x. Set

$$\mathcal{G} := \{g \in G \mid d_H(g(\bar{F}_i), \bar{F}_i) < D \text{ for } i = 1, 2\}.$$

Pick $x \in F_1$. Define a map $\pi : \mathcal{G}_0(x) \to \mathcal{G}(x)$ as follows. For each $y \in \mathcal{G}_0(x)$, pick $g_y \in \mathcal{G}_0$ such that $\psi(g_y)(x) = y$, choose $\sigma_y \in \Sigma$ such that $\sigma_y(\bar{F}_2)$ and $g^{-1}(\bar{F}_2)$ are contained in the same fiber, and set $\pi(y) := \psi(g\sigma_y)(x)$. The map π has bounded displacement, which implies that $\mathcal{G}(x)$ is a net in F_1. This proves the first assertion of Lemma.

Suppose that $G \curvearrowright Z$ is discrete. Then for each $x_1 \in \mathcal{G}(x)$, $\sigma \in \Sigma$, the set

$$\{y \in \mathcal{G}_0(x) \mid \pi(y) = x_1, \sigma_y = \sigma\}$$

has uniformly bounded cardinality. Hence π has multiplicity $< M|\Sigma| \leq Mm$ where M is independent of F_1, F_2, and x, and m is the number of fibers contained in $N_{RL}(\bar{F}_1)$. Since $\mathcal{G}_0(x)$ is an r_1-net in $\bar{F}_1$, the density estimate (6.3) follows.

□

Consider a pair of fibers (F_i, f_i), and choose a map $\rho : F_1 \to F_2$ which has "bounded displacement",

$$\delta := d(f_1, f_2 \circ \rho) < \infty. \tag{6.4}$$

Definition 6.5. Given an interval $I = [a, b] \subset F_1$, we define the *expansion of* I *under* ρ to be the quantity

$$E(I, \rho) := \frac{|\rho(a) - \rho(b)|}{|a - b|}$$

In what follows we will use the notation $|I|$ to denote the length of an interval $I \subset \mathbb{R}$.

Note that if $\rho' : F_1 \to F_2$ is another bounded displacement map, then the symmetric difference $[\rho(a), \rho(b)]\Delta[\rho'(a), \rho'(b)]$ has length bounded independent of a, b, and hence the ratio of the quantities $E(I, \rho)$ and $E(I, \rho')$ tends to 1 as $|I|$ tends to infinity.

Let $(F_1, f_1), (F_2, f_2)$ be a pair of fibers, and let ρ be a map satisfying (6.4).

Lemma 6.6. *There is a constant C which is independent of the choice of fibers and the map ρ, such that when I and I' are sufficiently long intervals, we have*

$$\frac{1}{C}E(I,\rho) \leq E(I',\rho) \leq CE(I,\rho). \tag{6.7}$$

Proof. The idea is to use "translates" of the image $\rho(I)$ of an interval I to get a controlled multiplicity covering of the image $\rho(J)$ of a longer interval J; this allows one to compare expansion factors.

Consider the collection $\mathcal{G}$ of elements of G determined by the pair of fibers F_1, F_2 as in Lemma 6.2. Each $g \in \mathcal{G}$ determines a quasi-isometry $\psi_{F_i}(g) : F_i \to F_i$, $i = 1, 2$, and the quasi-isometry constants of these maps are bounded independently of g and the choice of fibers F_1, F_2. Note that

$$d(\rho \circ \psi_{F_1}(g), \psi_{F_2}(g) \circ \rho) \leq C_1 \tag{6.8}$$

where C_1 is independent of $g \in \mathcal{G}$.

Let $\mathcal{C}$ be the collection of intervals of the form $[\psi_{F_1}(g)(a), \psi_{F_1}(g)(b)] \subset F_1$, where g ranges over $\mathcal{G}$ and $I = [a, b]$. The collection of left endpoints of the intervals in $\mathcal{C}$ is a net in F_1 by Lemma 6.2. For each interval $\hat{I} = [\hat{a}, \hat{b}] \in \mathcal{C}$, we have

$$|\hat{I}| \simeq |I| \quad \text{and} \quad |\rho(\hat{b}) - \rho(\hat{a})| \simeq |\rho(b) - \rho(a)| = E(I,\rho)|I| \tag{6.9}$$

provided $|I|$ is sufficiently big, since the family $\{\psi_{F_1}(g) : g \in \mathcal{G}\}$ has quasi-isometry constants (L, A) independent of F_1, F_2, and ρ is a quasi-isometry. Again using the fact that ρ is a quasi-isometry, we can find a constant C_2 such that if $\{x_1, x_2, x_3\} \subset F_1$ and $d(x_i, x_j) > C_2$ for $i \neq j$, then x_2 lies between x_1 and x_3 if and only if $\rho(x_2)$ lies between $\rho(x_1)$ and $\rho(x_3)$. When I is sufficiently long, we may choose a subcollection $\mathcal{C}_1 := \{[a_j, b_j]\}_{j \in \mathbb{Z}}$ of $\mathcal{C}$ (still covering $\mathbb{Z}$), such that for all $i \in \mathbb{Z}$,

$$a_i + C_2 < b_{i-1} < a_{i+1} - C_2 \quad \text{and} \quad b_{i-1} + C_2 < a_{i+1} < b_i - C_2. \tag{6.10}$$

Pick an interval $J = [c, d] \subset F_1$. Let $j_1 := \max\{j \in \mathbb{Z} \mid a_j + C_2 < c\}$ and $j_2 := \min\{j \in \mathbb{Z} \mid b_j - C_2 > d\}$, and set

$$\mathcal{C}_J := \{[a_j, b_j] : [a_j, b_j] \in \mathcal{C}_1, \quad j_1 \leq j \leq j_2\}.$$

Then $\mathcal{C}_J$ covers J with multiplicity ≤ 2, and

$$\sum_{j=j_1}^{j_2} |b_j - a_j|$$

is comparable to $|J|$ when $|J|$ is large. Note that the collection

$$\rho_*(\mathcal{C}_J) := \{[\rho(a_j), \rho(b_j)] \mid [a_j, b_j] \in \mathcal{C}_J\}$$

covers $[\rho(c), \rho(d)]$ with multiplicity ≤ 2, by (6.10). Also,

$$\sum_{j=j_1}^{j_2} \rho(b_j) - \rho(a_j) \simeq E(I, \rho)|J|$$

is comparable to $\rho(d) - \rho(c)$ when $|J|$ is large. It follows that $E(I, \rho)$ is comparable to $E(J, \rho)$. Applying the same reasoning to I' yields that $E(I', \rho)$ is comparable to $E(J, \rho)$, hence (6.8) follows. □

Definition 6.11. Consider an ordered pair of fibers $((F_1, f_1), (F_2, f_2))$, which we will conflate with (F_1, F_2). The *fiber expansion factor* for (F_1, F_2) is defined to be

$$E(F_1, F_2) := \limsup_{|I| \to \infty} E(I, \rho)$$

where $\rho : F_1 \to F_2$ is any bounded displacement map. The remark after Definition 6.5 implies that this is independent of ρ.

The next corollary follow from Lemma 6.6:

Corollary 6.12. *1. Given three fibers F_1, F_2, F_3, we have*

$$\frac{1}{C^2} E(F_1, F_2) E(F_2, F_3) \leq E(F_1, F_3) \leq E(F_1, F_2) E(F_2, F_3).$$

2. There is a constant C' such that if $(F_1, f_1), (F_2, f_2) \in \mathcal{F}$, $g \in G$, then

$$\frac{1}{C'} E(F_1, F_2) \leq E(g_* F_1, g_* F_2) \leq C' E(F_1, F_2).$$

Therefore we get

Corollary 6.13. *Pick a fiber F_0; for elements $g \in G$ consider the function*

$$l(g) = \log E(F_0, g_* F_0). \tag{6.14}$$

Then $l : G \to \mathbb{R}$ is a quasi-homomorphism.

6.2. **Coarse fibrations with amenable base.** Recall that a bounded geometry metric space S is called *amenable* if it admits a c-Følner sequence for each c. Recall that for a subset $D \subset S$ the set $\partial_c D$ consists of points in $S \setminus D$ which are within distance $\leq c$ from D.

Proposition 6.15. *If the fiber expansion function of $\mathcal{F}$ is bounded and the base $Y^{(0)}$ is amenable, then Z is amenable as well.*

Proof. Let $p : Z \to Y^{(0)}$ be the projection (recall that Z is the disjoint union of the fibers $\bar{F} \in \bar{\mathcal{F}}$). Let $D_k \subset Y^{(0)}$ be a sequence with $\frac{|\partial_c D_k|}{|D_k|} \to 0$.

Pick $(F, f) \in \mathcal{F}$. Choose $v_F : Z \to F$ so that $f \circ v_F : Z \to Z$ is a nearest point projection to $\bar{F}$. For each T, let $I_T \subset F$ be an interval of length T . For fixed k, consider the family of intersections $E_{k,T} := p^{-1}(D_k) \cap v_F^{-1}(I_T)$, where $T > 0$. Note that

$$\limsup_{T \to \infty} \frac{|\partial_c E_{k,T}|}{|E_{k.T}|} \leq C \frac{|\partial_c D_k|}{|D_k|},$$

where C is independent of k; so we can obtain a Følner sequence of the form E_{k,T_k} for an appropriately chosen sequence T_k. □

Remark 6.16. It is tempting to try to prove that X has polynomial growth provided Y has polynomial growth or is doubling. However, it seems that to prove this directly one would need stronger control over the nearest point maps between fibers.

6.3. **Coarse fibrations with subexponentially growing base.** Consider the fibration of $\mathbb{H}^2$ by horocycles asymptotic to $\xi \in \partial_\infty \mathbb{H}^2$. This fibration is invariant under the stabilizer of ξ in $PSL(2, \mathbb{R})$, which acts transitively on $\mathbb{H}^2$. If we discretize this example we get a coarse fibration of $\mathbb{H}^2$ by copies of $\mathbb{Z}$ whose base is quasi-isometric to $\mathbb{Z}$, with unbounded fiber expansion function. Examples like this are incompatible with a *discrete* quasi-action, however.

Proposition 6.17. *Assume that $G \curvearrowright Z$ is a discrete quasi-action, and $Y^{(0)}$ has subexponential growth. Then the fiber expansion function $E(\cdot, \cdot)$ of the coarse fibration $\mathcal{F}$ is bounded.*

Proof. Suppose that the function E is unbounded; therefore the quasi-homomorphism l defined in (6.14) is unbounded as well. Thus for suitable $g \in G$ and $\alpha_0 > 0$ we have

$$l(g^k) \leq -\alpha_0 k$$

for all $k > 0$. So there is an $\alpha > 0$ such that for all R_0, there are fibers F_1, $F_2 \in \mathcal{F}$ such that $d(\bar{F}_1, \bar{F}_2) = R \geq R_0$ and

$$E(F_1, F_2) \leq e^{-\alpha R}. \tag{6.18}$$

Take a bounded displacement function $\bar{\rho} : \bar{F}_1 \to \bar{F}_2$, then there exists a map $\rho : F_1 \to F_2$ such that $\bar{\rho} \circ f_1 = f_2 \circ \rho$.

Pick $x \in F_1$ and let $\mathcal{G}$, L be as in Lemma 6.2. The discreteness of $G \curvearrowright Z$ implies that the map $\mathcal{G} \to \mathcal{G}(x)$ (respectively $\mathcal{G} \to \mathcal{G}(\rho(x))$) given by $g \mapsto \psi_{F_1}(g)(x)$ (respectively $g \mapsto \psi_{F_2}(g)(\rho(x))$) has multiplicity bounded independently of F_1, F_2. Therefore, when $I = [a, b] \subset F_1$ is a sufficiently long interval we have

$$|\mathcal{G}(x) \cap [a, b]| \simeq |\mathcal{G}(\rho(x)) \cap [\rho(a), \rho(b)]|. \tag{6.19}$$

By Lemma 6.2, when $I = [a, b]$ is sufficiently long, we have

$$|\mathcal{G}(x) \cap [a, b]| \gtrsim \frac{b-a}{m} \tag{6.20}$$

where m is the number of fibers contained in $N_{LR}(\bar{F}_1)$. If we choose F_1, F_2 such that (6.18) holds, for long intervals $I = [a, b] \subset F_1$, by applying the estimates (6.19), (6.18) and (6.20) we obtain (for certain constants C_1, C_2):

$$C_1|\mathcal{G}(\rho x) \cap [\rho(a), \rho(b)]| \le \rho(b) - \rho(a) \le 2(b-a)e^{-\alpha R} \le C_2 m e^{-\alpha R}|\mathcal{G}(x) \cap [a, b]|. \tag{6.21}$$

When R is large, $m \ll e^{\alpha R}$ (recall that $Y^{(0)}$ has subexponential growth), so (6.21) contradicts the fact that the cardinalities $|\mathcal{G}(\rho x) \cap [\rho(a), \rho(b)]|$ and $|\mathcal{G}(x) \cap [a, b]|$ are comparable. □

By combining Propositions 6.15, 6.17 and using the fact that each bounded geometry metric space of subexponential growth is amenable (see e.g. Lemma 2.8) we obtain:

Corollary 6.22. *If $Y^{(0)}$ has subexponential growth and $G \curvearrowright Z$ is discrete then the fiber expansion function E is bounded and Z is amenable.*

6.4. **The homomorphism from G to $\mathrm{QI}(F)$.** Recall that for each $(F, f) \in \mathcal{F}$, we have a degenerating quasi-action $G \overset{\psi}{\curvearrowright} F$, where for each $g \in G$, we obtain the map $\psi(g) : F \to F$ by post-composing the composition $F \xrightarrow{f} Z \xrightarrow{g} Z$ with $v_F : Z \to F$, see Lemma 3.8. Let $(g_*F, f_g) \in \mathcal{F}$ denote the fiber for which $\overline{g_*F} \supset g(\bar{F})$. Then there is a quasi-isometry $h : F \to g_*F$, with constants independent of F and g, such that $d(g \circ f, f_g \circ h) < C$ where C is independent of F and g (Lemma 3.7). Hence $\psi(g) := v_F \circ g \circ f$ is at bounded distance from the composition $v_F \circ f_g \circ h$. The composition $\rho := v_F \circ f_g : g_*F \to F$ satisfies the inequality (6.4). This means that when x, $y \in F$ are far apart, then

$$\begin{aligned} d(\psi(g)(x), \psi(g)(y)) &\simeq d((v_F \circ f_g) \circ h(x), (v_F \circ f_g) \circ h(y)) \\ &\simeq E(g_*F, F)d(h(x), h(y)) \simeq E(g_*F, F)d(x, y). \end{aligned}$$

Or, to put it another way, for every $g \in G$, there are constants L and A, where L is independent of g and F, so that

$$L^{-1}E(gF,F)d(x,y) - A \leq d(\psi(g)(x), \psi(g)(y)) \leq LE(gF,F)d(x,y) + A \tag{6.23}$$

for points $x, y \in F$ which are sufficiently far apart. So, modulo rescaling the metric on the target by the factor $E(g_*F, F)$, the map $\psi(g)$ defines a quasi-isometry with uniformly controlled Lipschitz constant (but uncontrolled additive constant). We recall that if ω is a nonprincipal ultrafilter on $\mathbb{N}$, $\phi : W \to W'$ is an (L, A) quasi-isometry between metric spaces, $w_k \in W$, $w'_k \in W'$, and $\lambda_k \in \mathbb{R}_+$ are sequences with $\lambda_k \to \infty$ where $\omega\text{-}\lim_k \frac{1}{\lambda_k} d(\phi(w_k), w'_k) < \infty$ then ϕ induces an L-bilipschitz homeomorphism ϕ_ω between the asymptotic cones $\omega\text{-}\lim(\frac{1}{\lambda_k}W, w_k)$ and $\omega\text{-}\lim(\frac{1}{\lambda_k}W', w'_k)$. Hence if for some sequences $x_k \in F$ and $\lambda_k \in \mathbb{R}_+$ we have $\omega\text{-}\lim_k \frac{1}{\lambda_k} d(\psi(g)(x_k), x_k) < \infty$ for each element g of a generating set for G, then we get an induced homomorphism ψ_ω from G to the group of bilipschitz homeomorphisms of $F_\omega \equiv \mathbb{R}$. Equation 6.23 implies that for such a homomorphism ψ_ω, and each $g \in G$, we have

$$L^{-1}E(gF,F)d(x,y) \leq d(\psi_\omega(g)(x), \psi_\omega(g)(y)) \leq LE(gF,F)d(x,y), \tag{6.24}$$

which means that ψ_ω is an action by homeomorphisms which are all uniformly bilipschitz modulo rescaling of the target metric (by a factor which depends on the group element).

6.5. **The case when the homomorphism $G \to \mathrm{QI}(F)$ is trivial.** Assume the quasi-action $G \curvearrowright Z$ is discrete and cobounded, in particular, the group G is finitely generated. Pick a fiber $F \in \mathcal{F}$.

Lemma 6.25. *Suppose the homomorphism $G \to \mathrm{QI}(F)$ induced by the quasi-action $G \curvearrowright F$ is trivial, or equivalently, suppose $d(\psi(g), \mathrm{id}_F) < \infty$ for every $g \in G$. Then for any $x \in \bar{F}$, there is an r such that $N_r(Z_G(x)) \supset \bar{F}$, where Z_G is the center of G. In particular Z_G is infinite.*

Proof. Let $g_1, ..., g_n$ be generators of G. For each $i = 1, ..., n$ the quasi-isometry $\psi(g_i) : F \to F$ has bounded displacement.

Pick a point $x \in \bar{F}$. Since the quasi-action $G \curvearrowright Z$ is cobounded, there exists a subset $H \subset G$ such that the sets $H(x)$ and $\bar{F}$ are at finite Hausdorff distance. Consider the set of commutators

$$K := \{[g_1, h], h \in H\}.$$

Since $\psi(g_1)$ has bounded displacement, there is a constant $C_1 < \infty$ so that for each $k \in K$ we have $d(k(x), x) < C_1$. Hence discreteness of the quasi-action $G \curvearrowright Z$ implies that there is an element $h_1 \in H$ and a subset $J_1 \subset H$ so that

1. $[g_1, h_1] = [g_1, h]$, for each $h \in J_1$.
2. The sets $J_1(x)$ and $\bar{F}$ are Hausdorff-close.

Set $H_1 := h_1^{-1} J_1$. The first property implies that $[g_1, h_1^{-1} h] = 1$ for each $h \in J_1$, i.e.

$$[g_1, g] = 1, \forall g \in H_1.$$

The second property implies that $d_H(H_1(x), \bar{F}) < \infty$.

We now repeat the above argument using H_1 instead of H and proceed inductively. After n steps we get a subset $H_n \subset G$ so that:

3. $[g_i, h] = 1$, for each $h \in H_n$, $i = 1, ..., n$.
4. $d_H(H_n(x), \bar{F}) < \infty$.

Hence the center Z_G of the group G contains the infinite subset H_n and the assertion of the proposition follows. □

6.6. **Bounded expansion functions.** Suppose the expansion function $E(\cdot, \cdot)$ is bounded.

Lemma 6.26. *If the homomorphism $G \to \mathrm{QI}(F)$ is nontrivial, $F \in \mathcal{F}$, then there a nontrivial isometric action $G \curvearrowright \mathbb{R}$. If, moreover, the image of $G \to \mathrm{QI}(F)$ consists of orientation preserving quasi-isometries, then there is a nontrivial homomorphism $G \to \mathbb{R}$.*

Proof. Let $\{g_1, \dots, g_n\} \subset G$ be a generating set. For each $x \in F$ define

$$\delta(x) := \max\{d(x, \psi(g_i)(x)), i = 1, ..., n\}.$$

By assumption, the quasi-isometry $\psi(g_i) : F \to F$ has unbounded displacement for some $1 \le i \le n$. Therefore there is a sequence of points $x_j \in F$ such that

$$\limsup_{j \to \infty} \delta(x_j) = \infty.$$

Let ω be a nonprincipal ultrafilter. Then (after renumbering the generators g_i and passing to a subsequence of (x_j) if necessary) we may assume:

$$\lambda_j := \delta(x_j) = d(x_j, \psi(g_1)(x_j))$$

for all j. Thus

$$\lim_{j \to \infty} \lambda_j = \infty.$$

We now consider the ultralimit

$$\omega\text{-}\lim_j (F, \frac{1}{\lambda_j} d_F, x_j) = F_\omega.$$

Since the fiber expansion function is bounded, the quasi-action $\psi : G \curvearrowright F$ has bounded multiplicative quasi-isometry constants, see the inequality (6.23). Therefore after passing to the ultralimit we get an action $\psi_\omega : G \curvearrowright F_\omega \cong \mathbb{R}$ by uniformly bilipschitz homeomorphisms. Hence the action ψ_ω is conjugate to an action of G on $\mathbb{R}$ by isometries (see for instance [6]). The action of g_1 is nontrivial $(\psi_\omega(g_1)(x_\omega), x_\omega) = 1$ due to our choice of scale factors. When $G \to \mathrm{QI}(F)$ is orientation preserving, so is ψ_ω, and we get a nontrivial isometric action $G \curvearrowright \mathbb{R}$ by translations. □

We summarize our results in the following

Corollary 6.27. *Suppose that $\mathcal{F}$ is a parameterized coarse fibration of Z by lines so that the base $Y^{(0)}$ has polynomial growth and $G \curvearrowright Z$ is a discrete cobounded quasi-action which preserves the fibration $\bar{\mathcal{F}}$ and orientation on the fibers. Then:*

1. Z is amenable.

2. (a) Either G contains an infinite cyclic central subgroup Z whose orbit is Hausdorff-close to a fiber of $\bar{\mathcal{F}}$, or

(b) G admits a nontrivial homomorphism to $\mathbb{Z}$.

7. 3-manifolds coarsely fibered by lines

In this section we will prove the main theorem of our paper, Theorem 1.3. Suppose that M is a closed 3-manifold whose universal cover X admits a G-invariant parameterized coarse fibration

$$\mathcal{F} = \{(F_j, f_j)\}_{j \in \mathcal{J}}$$

by lines, where $G = \pi_1(M)$.

Suppose G has more than one end. Then there is a compact set $K \subset X$ and a pair C_1, C_2 of distinct unbounded components of $X \setminus K$. Suppose $\bar{F}_0 \in \bar{\mathcal{F}}$ is a fiber lying in a single component C of $X \setminus K$. Since $d_H(\bar{F}, \bar{F}_0) < \infty$ for every $\bar{F} \in \bar{\mathcal{F}}$, the portion of $\bar{F}$ lying outside C has bounded diameter. Using the fact that $\bar{F}$ is the image of a uniform embedding $\mathbb{R} \to X$ with controlled distortion, we conclude that $\bar{F} \setminus C$ has *uniformly* bounded diameter (for all coarse fibers $\bar{F}$ of $\bar{\mathcal{F}}$). Pick $i \in \{1, 2\}$ so that $C_i \neq C$. We then can find fibers passing through points $p \in C_i$ arbitrarily far from K, which is a contradiction. Therefore every fiber $\bar{F}$ must intersect at least two components of $X \setminus K$, which forces $\bar{F}$ to pass within uniform distance of K. It follows that X is at finite Hausdorff distance from every fiber, and so G is 2-ended. This implies that $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_2 \star \mathbb{Z}_2$, and in both cases M is homotopy-equivalent to a Seifert manifold.

Henceforth we will assume that G is 1-ended, which implies that X is contractible. Since we are interested in describing M up to homotopy-equivalence, we can assume that the manifold M is irreducible. If a manifold M is finitely covered by a Seifert manifold or a Solv-manifold, then M is itself Seifert or Solv-manifold (see [15] in Seifert case and [4] in the Solv-case); hence we a free to pass to finite index subgroups of G when convenient. Therefore, we pass to an index 2 subgroup in G that *preserves the orientation* of the fibers of $\mathcal{F}$, i.e. each element of G acts trivially on the set of ends of each fiber of $\mathcal{F}$.

We note that X has a natural structure of a uniformly contractible metric cell complex, the action $G \curvearrowright X$ is a properly discontinuous, free[2], cocompact and preserves the structure of a metric cell complex on X. Hence (see Proposition 4.1) there exists a uniformly contractible metric cell complex Y, the *coarse base* of the coarse fibration $\mathcal{F}$. Note also that the (quasi)-action $G \curvearrowright X^{(0)}$ projects to a cobounded quasi-action $G \curvearrowright Y^{(0)}$, see Remark 3.3. According to Corollary 5.4, the complex Y is a quasi-plane.

Our goal is to analyze the group G and the geometry of the coarse fibration depending upon the coarse geometry of the space Y and the dynamics of the quasi-action $G \curvearrowright Y^{(0)}$.

We will use Theorem 1.2 from [10]:

Theorem 7.1. *If Y is a quasi-plane and $G \curvearrowright Y$ is a cobounded action then one of the two cases can occur:*

Case 1. *$Y^{(0)}$ is Gromov-hyperbolic. In this case there exists an quasi-isometry $h : Y^{(0)} \to \mathbb{H}^2$ which conjugates the quasi-action of G to an isometric cobounded action $G \curvearrowright \mathbb{H}^2$.*

Case 2. *$Y^{(0)}$ has is doubling, and hence has polynomial growth. If Y is quasi-isometric to a finitely generated group Q then Q is virtually abelian of rank 2.*

7.1. **The hyperbolic case.** Assume that Y is a Gromov-hyperbolic, in this case (by Theorem 7.1) there exists a cobounded isometric action of G on $\mathbb{H}^2$ (which is quasi-isometrically conjugate to the quasi-action $G \curvearrowright Y$). The action $G \curvearrowright \mathbb{H}^2$ factors through a homomorphism

$$\phi : G \to \hat{G} \subset \mathrm{Isom}(\mathbb{H}^2).$$

Below is the discussion of the three possible types of isometric actions $\hat{G} \curvearrowright \mathbb{H}^2$.

Lemma 7.2. *Each cobounded isometric action $\Gamma \curvearrowright \mathbb{H}^2$ belongs to one of the following types:*

1) Γ is dense in $PSL(2, \mathbb{R})$.

2) Γ is a discrete cocompact lattice in $\mathrm{Isom}(\mathbb{H}^2)$.

3) Γ fixes a point $\xi \in \partial_\infty \mathbb{H}^2$ and the closure of Γ in $\mathrm{Isom}(\mathbb{H}^2)$ *is a solvable nonabelian group.*

Proof. Since the connected component $\overline{\Gamma}^0$ of the closure $\overline{\Gamma}$ in $\mathrm{Isom}(\mathbb{H}^2)$ is a connected Lie subgroup (which is trivial iff Γ is discrete and which acts transitively on $\mathbb{H}^2$ otherwise), $\overline{\Gamma}^0$ is either $\{1\}$, or a solvable nonabelian, or is equal to $PSL(2, \mathbb{R})$. □

[2]Most of the discussion remains valid in the case when G has torsion.

Case 1a. The action $\hat{G} \curvearrowright \mathbb{H}^2$ is discrete. Then $\hat{G}$ is a cocompact Fuchsian group, which therefore has virtual cohomological dimension 2. Since the cohomological dimension of G is 3, the homomorphism ϕ has to have a nontrivial kernel. Moreover, each orbit of this kernel is Hausdorff-close to a fiber $\bar{F}$ in the coarse fibration $\bar{\mathcal{F}}$. Hence $Ker(\phi)$ is quasi-isometric to $\mathbb{Z}$, which in turn implies that $Ker(\phi)$ is isomorphic to $\mathbb{Z}$ (recall that G is torsion-free). Hence the fundamental group G of the 3-manifold M fits into a short exact sequence:

$$1 \to \mathbb{Z} \to \pi_1(M) \to \hat{G} \to 1.$$

Thus M is a Seifert manifold with hyperbolic base-orbifold (whose fundamental group is isomorphic to $\hat{G}$), see [15], and the fibers of $\bar{\mathcal{F}}$ are uniformly close to the lifts of the fibers of Seifert fibration from M to X.

In the following two cases the action $\hat{G} \curvearrowright \mathbb{H}^2$ is nondiscrete; hence the quasi-action $G \curvearrowright Y^{(1)}$ is nondiscrete as well, i.e. there exists $p \in Y^{(0)}$ and sequences of $g_i \in G$ and numbers $R_i \to \infty$ such that

$$d(g_i|_{B_{R_i}(p)}, id) \leq Const, \quad B_{R_i}(p) \subset Y^{(1)},$$

but each $g_i : Y^{(1)} \to Y^{(1)}$ does not have bounded displacement. In this case the *kernel* of the quasi-action $G \curvearrowright Y$ has to be trivial since the action $G \curvearrowright X$ is discrete and $Y^{(1)}$ is Gromov-hyperbolic (see section 2.1). In particular, the kernel of the homomorphism $\phi : G \to \hat{G}$ is trivial.

Case 1b. $\hat{G}$ is dense in $PSL(2, \mathbb{R})$. We claim that this case cannot occur. Since $\hat{G}$ is dense in $PSL(2, \mathbb{R})$, it contains a nontrivial elliptic element $\hat{g}$ and it also contains a sequence of elements $\hat{h}_i$ which converge to $1 \in PSL(2, \mathbb{R})$. Let $g, h_i \in G$ be elements which map (via ϕ) to $\hat{g}$ and $\hat{h}_i$ respectively. There exists $r \in \mathbb{R}$ such that for each $m \in \mathbb{Z}$, $g^m(\bar{F}) \subset N_r(\bar{F})$. Hence we obtain an orientation-preserving quasi-action

$$\langle g \rangle \curvearrowright F = \mathbb{R}$$

and it follows that g acts on F with bounded displacement.

By taking conjugates $g_i := h_i g h_i^{-1}$, we get an infinite collection of distinct elements $\{g_i : i \in \mathbb{N}\}$ of G so that for each $n \in \mathbb{Z}$, $g_i(\bar{F})$ is contained in $N_R(\bar{F})$ where $R \in \mathbb{R}_+$ and $\bar{F} \in \bar{\mathcal{F}}$ are independent of i. We note that since all g_i are pairwise conjugate to g, there exists $C < \infty$ such that $d(x, g_i(x)) < C$ for each $x \in \bar{F}$ and $i \in \mathbb{N}$. This contradicts discreteness of the action of G on X.

Case 1c. $\hat{G}$ is solvable. Hence, since $\mathrm{Ker}(\phi : G \to \hat{G})$ is trivial, G is virtually solvable itself. Therefore, according to the classification of Haken manifolds with solvable fundamental groups (see [5]), M is modeled on one of the three geometries: $\mathbb{E}^3$, Nil, Solv.

7.2. **The case of doubling base. Case 2.** $Y^{(0)}$ has polynomial growth. Then according to Corollary 6.27, the group G is amenable and either

2a. G has infinite center Z_G, or

2b. M is Haken.

In the first case we can use the solution of Seifert conjecture, in the case at hand (i.e., when G is amenable) it follows from G. Mess' work [14] that M is a Seifert manifold modeled on $\mathbb{E}^3$ or Nil geometry. As an alternative we can use the following trick. Since the infinite group Z_G is torsion-free, it contains an infinite cyclic subgroup $Z \subset Z_G$. The Z subgroup determines a coarse fibration $\mathcal{L}$ of X by lines (a priori this coarse fibration is not Hausdorff-close to $\bar{\mathcal{F}}$).

We now argue as before: form the coarse base Y' of the coarse fibration $\mathcal{L}$. The quasi-action $G \curvearrowright X$ descends to a quasi-action $G \curvearrowright Y'$ and the complex Y' is a quasi-plane. Now however the space $Y'^{(0)}$ is quasi-isometric to a Cayley graph of the quotient group $G' = G/Z$ and the quasi-action $G' \curvearrowright Y'^{(0)}$ is discrete and cocompact.

If $Y'^{(0)}$ is quasi-isometric to the hyperbolic plane then we get a homomorphism $G' \to \mathrm{Isom}(\mathbb{H}^2)$ whose image is a discrete cocompact subgroup: this contradicts amenability of G. If $Y'^{(0)}$ has polynomial growth then the group G' has polynomial growth and therefore is virtually nilpotent. This implies that the group G is virtually nilpotent as well and hence the manifold M is modeled on $\mathbb{E}^3$ or Nil. ☐

In the case 2b we use

Lemma 7.3. *Each closed Haken 3-manifold with amenable fundamental group is geometric, and hence must be modeled on* $\mathbb{E}^3$*,* Nil*, or* Solv.

Proof. We give two different proofs.

First proof. By Thurston's Geometrization theorem for Haken manifolds, our manifold M admits a decomposition along genus 1 surfaces into geometric components which are each diffeomorphic to quotients of model spaces by lattices or are covered by $[0,1] \times T^2$. Suppose first that M is not geometric. If all components in the geometric decomposition are covered by $[0,1] \times T^2$ then the entire manifold M is actually geometric and we obtain a contradiction.

Otherwise at least one component M_i in the geometric decomposition has the form X/Γ_i, where Γ_i is a nonuniform lattice. But nonuniform lattices occur only in the geometries $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL(2,\mathbb{R})}$, all of whose lattices are nonamenable. Subgroups of amenable groups are amenable, so this is impossible. If M is geometric the same argument implies that M is modeled on $\mathbb{E}^3$, Nil, or Solv.

Second proof. We let $G := \pi_1(M)$. Consider a two-sided, embedded, incompressible surface $S \subset M$. Since G is amenable, S has genus 1. Let $G \curvearrowright T$ be the action on the Bass-Serre tree corresponding to the decomposition of G along $\pi_1(S)$. Since G

is amenable, either T is a line, or G must fix a point in the ideal boundary of T (otherwise G will contain a free nonabelian subgroup).

In the first case, after passing to an index two subgroup if necessary, the action $G \curvearrowright T$ will be by translations, and hence we get an exact sequence $1 \to H \to G \to \mathbb{Z} \to 1$ where $H = \pi_1(S)$. Therefore, by the Stallings-Waldhausen theorem, M is homeomorphic to a surface bundle over a circle with fiber S. Thus the manifold M is modeled on $\mathbb{E}^3$, Nil, or Solv.

We claim that the second case is impossible unless T is a line. Let $v \in T$ be a vertex. The vertex stabilizer G_v must fix the incident edge e which points towards the fixed point at infinity. Hence the vertex group G_v coincides with $G_e \simeq \pi_1(S)$. Hence the stabilizers of all edges incident to v are of finite index in G_v (all groups are virtually $\mathbb{Z}^2$). If there were more than two edges incident to v, we could pass to a finite index subgroup G' of G_v which fixed at least two of them, and then the quotient of the vertex space by G' would have two boundary components whose inclusion is a homotopy equivalence, which implies that the difference of (the fundamental classes of the) two boundary components is null homologous, and hence there can be no other boundary components. This means that T is a line. $\square$

Thus in the both cases 2a, 2b, the manifold M is geometric modeled on $\mathbb{E}^3$, Nil, or Solv. This concludes the proof of the main theorem. $\square$

8. Classification of coarse fibrations of geometric 3-manifolds

In this section we classify invariant coarse fibrations of the universal covers of closed irreducible 3-manifolds M. By the main theorem we already know that every such M must be either Seifert or Solv-manifold. We recall several constructions of coarse fibrations:

1. Suppose that M is a Seifert manifold with noncompact universal cover X; consider the lift $\bar{\mathcal{L}}$ of the Seifert fibration on M to X. Then $\bar{\mathcal{L}}$ is a coarse fibration of X by lines.

2. Let $h : T^2 \to T^2$ be an Anosov affine mapping. Then there exists two h-invariant foliations $\mathcal{G}$ of T^2 by affine geodesics. Let M be the mapping torus of h. Then each foliation $\mathcal{G}$ yields a 1-dimensional lamination $\hat{\mathcal{L}}$ on M. Let $\bar{\mathcal{L}}$ denote the lift of $\hat{\mathcal{L}}$ to the universal cover X of M. Then $\bar{\mathcal{L}}$ is a coarse fibration of X by lines.

3. Let M be a Euclidean 3-manifold, i.e. the quotient of $X = \mathbb{E}^3$ by a torsion-free lattice G. Consider a G-invariant family $\bar{\mathcal{L}}$ of parallel lines in X. Then $\bar{\mathcal{L}}$ is a coarse fibration of X by lines.

We will refer to these examples as *standard coarse fibrations by lines.* The main result of this section is

Theorem 8.1. *Suppose that M is a Seifert manifold or* Solv*-manifold with universal cover X and $\bar{\mathcal{F}}$ is a $\pi_1(M)$-invariant coarse fibration of X by lines. Then $\bar{\mathcal{F}}$ is Hausdorff-close to a standard coarse fibration $\bar{\mathcal{L}}$ of X by lines.*

In the case when the coarse base Y of the coarse fibration $\bar{\mathcal{F}}$ is hyperbolic (which includes the Solv-manifolds and Seifert manifolds with hyperbolic base), the proof of the main theorem immediately implies that $\bar{\mathcal{F}}$ is within finite distance from a standard coarse fibration $\bar{\mathcal{L}}$ by lines.

Thus we are done in the case when M is a Seifert manifold with hyperbolic base. However even in the case of Solv-manifolds, there is (an a priori) possibility that the coarse base Y of the coarse fibration $\bar{\mathcal{F}}$ has polynomial growth. Hence we are left with three classes of 3-manifolds M to consider:

(a) Euclidean manifolds.

(b) Nil-manifolds.

(c) Solv-manifolds.

Thus (after passing to a finite cover) we may assume that our manifold M is of the form K/G, where K is $\mathbb{R}^3$, the Heisenberg group, or the solvable connected Lie group $\mathrm{Isom}^o(\mathrm{Solv})$, and G is a torsion-free lattice in K.

The following lemma will be used repeatedly in the analysis.

Lemma 8.2. *Let $G \curvearrowright X$ be a cobounded quasi-action on a metric space, and let $\bar{\mathcal{F}}$ be a G-invariant coarse fibration. Then there is a constant D with the following property. For all $\bar{F} \in \bar{\mathcal{F}}$, x, $x' \in \bar{F}$, there are sequences $\{x_i\} \in \bar{F}$, $\{g_i\} \in G$ such that for all i,*

$$d_H(g_i\bar{F}, \bar{F}) < D \tag{8.3}$$

and

$$d(g_i x, x_i) < D, \quad d(g_i x', x_{i+1}) < D. \tag{8.4}$$

Proof. Set $x_0 := x$, $x_1 := x'$, and $g_0 = 1 \in G$. Using the coboundedness of $G \curvearrowright X$, pick $g_1 \in G$ such that $d(g_1x_0, x_1) < D_1$, where D_1 depends only on the quasi-action $G \curvearrowright X$. Then $d_H(g_1\bar{F}, \bar{F}) < D_2$ where D_2 depends only on the constants of the coarse fibration $\bar{\mathcal{F}}$. So we can pick $x_2 \in \bar{F}$ such that $d(x_2, g_1x_1) < D_2$. Proceeding by induction, we define g_i, x_{i+1} for $i > 0$. Define g_i, x_i for $i < 0$ inductively as follows. Assume g_k, x_k have been defined for $k > i$. Pick $g_i \in G$ such that $d(g_ix_1, x_{i+1}) < D_1$, and $x_i \in \bar{F}$ such that $d(x_i, g_ix_0) < D_2$. The sequences $\{x_i\}$, $\{g_i\}$ satisfy (8.3) and (8.4) with $D = \max(D_1, D_2)$. □

Corollary 8.5. *For every $x, x' \in \bar{F}$ the mapping $\iota : \mathbb{Z} \to X$ given by $i \mapsto x_i$ is $(2D + d(x, x'))$-Lipschitz.*

Lemma 8.6. *Suppose $G \curvearrowright \mathbb{R}^n$ is a cobounded quasi-action by translations, and $\bar{\mathcal{F}}$ is a G-invariant coarse fibration by lines. Then $\bar{\mathcal{F}}$ is at finite Hausdorff distance from an affine foliation by lines: there is a line $L \subset \mathbb{R}^n$ so that every fiber $\bar{F} \in \bar{\mathcal{F}}$ is at uniformly bounded Hausdorff distance from a line parallel to L.*

Proof. Pick $\bar{F} \in \bar{\mathcal{F}}$, $p \in \bar{F}$. For each $q \in \bar{F}$, we apply Lemma 8.2 with $x = p$ and $x' = q$ to obtain sequences $\{x_i(q)\} \in \bar{F}$, $\{g_i(q)\} \in G$ and constant D. Since G quasi-acts by translations, for all $q \in \bar{F}$, $i \in \mathbb{Z}$, we have $|(x_{i+1}(q) - x_i(q)) - (q - p)| < 2D$. This means that for all $i < j \in \mathbb{Z}$, $q \in \bar{F}$,

$$|(x_j(q) - x_i(q)) - (j - i)(q - p)| < 2D(j - i). \tag{8.7}$$

So when $|q - p| > 2D$, the map $i \mapsto x_i(q) \in \bar{F} \subset \mathbb{R}^n$ is bilipschitz:

$$|j - i|(|q - p| - 2D) < |x_j(q) - x_i(q)| < |j - i|(|q - p| + 2D),$$

which clearly implies that $\{x_i(q)\}$ is an ϵ-net in $\bar{F}$, for some $\epsilon = \epsilon(q)$. By applying the inequality (8.7) to $i = 0$ we get for $x = x_j(q)$:

$$|(x - p) - j(q - p)| < 2Dj, \tag{8.8}$$

and hence (provided that $|q - p| > 2D$)

$$|x - p| > j(|q - p| - 2D), \quad j < \frac{|x - p|}{|q - p| - 2D}. \tag{8.9}$$

Therefore by combining (8.8) with (8.9) we get:

$$d(x - p, j(q - p)) < 2Dj < \frac{2D|x - p|}{|q - p| - 2D}. \tag{8.10}$$

Let L_q be the line $\{p + t(q - p) \mid t \in \mathbb{R}\}$. By (8.10), we get that $\{x_i(q)\}$ is contained in the cone

$$\Omega_q := \left\{ x \in \mathbb{R}^n \mid \frac{d(x, L_q)}{d(x, p)} \leq \frac{2D}{|q - p| - 2D} \right\},$$

when $|q - p| > 2D$, and hence $\bar{F}$ is contained in the ϵ-neighborhood of Ω_q. Observe that the angle of the cone Ω_q converges to zero as $|q| \to \infty$. This implies that the family of subspaces $\mathbb{R}(q - p) \subset \mathbb{R}^n$ converges to a line $\mathbb{R}v$ as $|q| \to \infty$. Indeed, otherwise for some $q_1, q_2 \in \bar{F}$, the cones $\Omega_{q_1}, \Omega_{q_2}$ would have intersection only at p; it follows that

$$\operatorname{diam}(N_\epsilon(\Omega_{q_1}) \cap N_\epsilon(\Omega_{q_2})) \leq \epsilon,$$

where $\epsilon := \max(\epsilon(q_1), \epsilon(q_2))$. However the intersection $N_\epsilon(\Omega_{q_1}) \cap N_\epsilon(\Omega_{q_2})$ contains the fiber $\bar{F}$ whose diameter is infinite. Contradiction.

Let L be the line $\{p + tv \mid t \in \mathbb{R}\}$, the limit of the lines $L_q, |q| \to \infty$. Observe that for every divergent sequence $x_j \in \bar{F}$ the segments $\overline{px_j}$ converge to a ray in the line L. Indeed, for each $q \in \bar{F}$ as above, the limit of every convergent subsequence of segments $(\overline{px_j})_{j\in\mathbb{N}}$ would have to belong to the cone Ω_q. Since these cones converge to the line L as $|q|\to\infty$, the claim follows. Thus, in particular,

$$p+\lim_{j\to\infty}\frac{x_j(q)-p}{j}\in L. \tag{8.11}$$

We now claim that $\bar{F}\subset N_{2D}(L)$. If there exists $q\in\bar{F}\setminus N_{2D}(L)$, then (by the triangle inequality)

$$d(x_j(q),L)\geq d(p+j(q-p),L)-|(p+j(q-p))-x_j(q)|,$$

the latter is

$$\geq d(q,L)j-2Dj=j(d(q,L)-2D)$$

by (8.8). Hence

$$\liminf_{j\to\infty}\frac{d(x_j(q),L)}{j}>0$$

which is a contradiction with (8.11). This proves the claim.

Since $\bar{F}$ is a line which is quasi-isometrically embedded in $N_{2D}(L)$, we clearly have $d_H(\bar{F},L)<\infty$. Translation invariance then implies that every fiber in $\bar{\mathcal{F}}$ is within uniformly bounded Hausdorff distance from a translate of L. □

We now consider the case of Nil geometry. Let Nil denote the Heisenberg group endowed with a left invariant Riemannian metric. We will use the notation Z for the center of the group Nil, recall that Z is isomorphic to $\mathbb{R}$. Let $\pi:\mathrm{Nil}\to\mathbb{R}^2\simeq\mathrm{Nil}/Z$ be canonical epimorphism, and endow $\mathbb{R}^2$ with the submersion metric, so that π has Lipschitz constant 1. Let $G\curvearrowright\mathrm{Nil}$ be a cobounded quasi-action by left translations which preserves a coarse fibration $\bar{\mathcal{F}}$ of Nil by lines.

Lemma 8.12. *There is a constant C such that every fiber is a Hausdorff distance at most C from a coset of the center Z.*

Proof. Pick $\bar{F}\in\bar{\mathcal{F}}$, $p\in\bar{F}$. Let D be the constant given by Lemma 8.2 with $x=p$ and $x'=q$, and for every $q\in\bar{F}$, let $\{x_i(q)\}$, $\{g_i(q)\}$ be the sequences provided by that lemma. Set $y_i(q):=\pi(x_i(q))$.

We now suppose that $\mathrm{diam}(\pi(\bar{F}))>2D$, and will derive a contradiction. Pick $p,q\in\bar{F}$ such that $|\pi(p)-\pi(q)|>2D$.

The quasi-action $G\overset{\Psi}{\curvearrowright}\mathrm{Nil}$ covers a quasi-action $G\overset{\Psi_{\mathbb{R}^2}}{\curvearrowright}\mathbb{R}^2$ by translations. Hence we can repeat the calculations of the previous lemma, to get that for all $i<j\in\mathbb{Z}$,

$$|(y_j(q)-y_i(q))-(j-i)(\pi(q)-\pi(p))|<2D(j-i),$$

and

$$|y_j(q)-y_i(q)|\geq(j-i)(|\pi(q)-\pi(p)|-2D). \tag{8.13}$$

Since $|\pi(q)-\pi(p)| > 2D$, the map $i \mapsto y_i(q)$ is a bilipschitz embedding, by combining this with Corollary 8.5 we conclude that the map $\iota : i \mapsto x_i(q)$ is also a bilipschitz embedding. Observe that (8.13) also implies that $\mathrm{diam}(\pi(\bar{F})) = \infty$. Moreover, $\{x_i(q)\}$ is a net in $\bar{F}$, and $y_i(q)$ is a net in $\pi(\bar{F})$. Repeating arguments from Lemma 8.6, we conclude that $\pi(\bar{F}) \subset N_{2D}(L)$ for some line $L \subset \mathbb{R}^2$ passing through $\pi(p)$.

Now consider the net $\{x_i(q)\} \subset \bar{F}$ for some $q \in \bar{F}$ with $|\pi(q)-\pi(p)| > 2D$. Pick $g \in G$ such that $\Psi_{\mathbb{R}^2}(g)(L) \neq L$. Since $\Psi_{\mathbb{R}^2}(g) : \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, its displacement equals to some $\delta \in \mathbb{R}_+$, and hence (by (8.13)):

$$\begin{aligned} d(g(x_i(q)), x_j(q)) &\geq d(\Psi_{\mathbb{R}^2}(g)(y_i(q)), y_j(q)) \\ &\geq |j-i|(|\pi(q)-\pi(p)| - 2D) - \delta. \end{aligned} \tag{8.14}$$

On the other hand, the displacement function $\delta_g : \mathrm{Nil} \to \mathbb{R}$ of g is constant on fibers of π, and unbounded on $\pi^{-1}(L)$. Hence $d(g(x_i(q)), x_i(q))$ is unbounded in i. By assumption, $d_H(g(\bar{F}), \bar{F}) \leq c < \infty$, hence (since $\{x_i(q)\}$ is a net in $\bar{F}$) there exists $j = j(i)$ such that

$$d(g(x_i(q)), x_{j(i)}(q)) \leq c.$$

Since, $d(g(x_i(q)), x_i(q))$ is unbounded in i, the function $|j(i)-i|$ is unbounded as well. However then (8.14) implies that

$$\lim_{i\to\infty} d(g(x_i(q)), x_{j(i)}(q)) = \infty.$$

Contradiction. □

We now turn to Solv geometry. We view Solv as $\mathbb{R}^3$ endowed with the metric $h = (e^z)^2 dx^2 + (e^{-z})^2 dy^2 + dz^2$. Let $\mathbb{R}^2_{xz}$ and $\mathbb{R}^2_{yz}$ denote the xz and yz planes respectively, endowed with the hyperbolic metrics $(e^z)^2 dx^2 + dz^2$ and $(e^{-z})^2 dy^2 + dz^2$, respectively, so that both $\mathbb{R}^2_{xz}$ and $\mathbb{R}^2_{yz}$ are isometric to $\mathbb{H}^2$. The projections $\pi_{xz} : \mathrm{Solv} = \mathbb{R}^3 \to \mathbb{R}^2_{xz}$, $\pi_{yz} : \mathrm{Solv} = \mathbb{R}^3 \to \mathbb{R}^2_{yz}$ are 1-Lipschitz. Let π_z denote the projection of $\mathrm{Solv} = \mathbb{R}^3$ to the z-axis.

Lemma 8.15. *Let $G \overset{\Psi}{\curvearrowright}$ Solv be an isometric (cocompact) action of a lattice on* Solv *by left translations, and let $\bar{\mathcal{F}}$ be a G-invariant coarse fibration of* Solv. *Then $\bar{\mathcal{F}}$ is at finite Hausdorff distance from a foliation of* $\mathrm{Solv} \simeq \mathbb{R}^3$ *by lines parallel to the x-axis or by lines parallel to the y-axis.*

Proof. We first note that without loss of generality we can assume that the lattice G is generated by a lattice H acting on the Euclidean plane $\mathbb{R}^2_{xy}$ by translations and an isometry

$$\tau : (x, y, z) \to (e^{-1}x, ey, z+1).$$

Hence the foliation $\bar{\mathcal{L}}$ of Solv by lines parallel to the x-axis or y-axis is an example of a *standard coarse fibration.*

Pick $\bar{F} \in \bar{\mathcal{F}}$, $p \in \bar{F}$, and as before, for each $q \in \bar{F}$, we apply Lemma 8.2 to get sequences $\{x_i(q)\} \subset \bar{F}$, $\{g_i(q)\} \in G$.

Suppose that for some $q \in \bar{F}$, $|\pi_z(q) - \pi_z(p)| > 2D$. Since the map $\pi_z : \text{Solv} \to \mathbb{R}$ is equivariant with respect to a translation action $G \curvearrowright \mathbb{R}$, we can obtain estimates similar to (8.13) for the sequence $\{\pi_z(x_i(q))\} \in \mathbb{R}$, which in turn implies that the sequences $\{\pi_{xz}(x_i(q))\}_{i\in\mathbb{Z}}$, $\{\pi_{yz}(x_i(q))\}_{i\in\mathbb{Z}}$ are bilipschitz embeddings of $\mathbb{Z}$ in $\mathbb{R}^2_{xz}$ and $\mathbb{R}^2_{yz}$ respectively. Moreover, the mapping $i \mapsto \pi_z(x_i(q))$ is a quasi-isometry $\mathbb{Z} \to \mathbb{R}$. Recall that according to Morse Lemma, each (L, A)-quasi-geodesic in a hyperbolic space is within Hausdorff distance $\le C(L, A)$ from a geodesic. Therefore, by the Morse Lemma applied to the hyperbolic planes $\mathbb{R}^2_{xz}, \mathbb{R}^2_{yz}$, there are vertical lines $L_x \subset \mathbb{R}^2_{xz}$, $L_y \subset \mathbb{R}^2_{yz}$ such that

$$d_H(\pi_{xz}(\bar{F}), L_x) < \infty, \quad d_H(\pi_{yz}(\bar{F}), L_y) < \infty.$$

For all $g \in G$, we have $d_H(g(\bar{F}), \bar{F}) < \infty$, which forces $\Psi_{xz}(g)(L_x) = L_x$, $\Psi_{yz}(g)(L_y) = L_y$ for all $g \in G$, where $G \overset{\Psi_{xz}}{\curvearrowright} \mathbb{R}^2_{xz}$ is the action on $\mathbb{R}^2_{xz}$ covered by $G \overset{\Psi}{\curvearrowright} \text{Solv}$, and likewise $G \overset{\Psi_{yz}}{\curvearrowright} \mathbb{R}^2_{yz}$ is the action on $\mathbb{R}^2_{yz}$ covered by $G \overset{\Psi}{\curvearrowright} \text{Solv}$. Clearly this contradicts the cocompactness of $G \overset{\Psi}{\curvearrowright} \text{Solv}$.

Hence each fiber $\bar{F} \in \bar{\mathcal{F}}$ lies in the $2D$ neighborhood of a horizontal plane. Therefore $\bar{\mathcal{F}}$ induces a coarse fibration of each horizontal plane. The stabilizer of the plane $\{z = c\}$ in G is the subgroup H which acts cocompactly on $\{z = c\}$ by translations. Applying Lemma 8.6, we conclude that the induced coarse fibration of $\{z = c\}$ is at finite Hausdorff distance from a fibration by parallel lines. Unless the lines are parallel to the x-axis or y-axis, an element $g \in G \setminus H$ will move a fiber $\bar{F}$ to a set at infinite Hausdorff distance from $\bar{F}$, contradicting the G-invariance of $\bar{\mathcal{F}}$. ☐

9. Coarse fibrations of hyperbolic groups

The main result of this section is the following

Theorem 9.1. *Suppose that G is a Gromov hyperbolic group which admits a G-invariant coarse fibration $\bar{\mathcal{F}}$ by lines. Then G is* elementary, *i.e. it is either finite or it is commensurable to $\mathbb{Z}$.*

Proof. We will identify coarse fibration of G with a coarse fibration of its Cayley graph X. Without loss of generality we may assume that G preserves orientation of the fibers of $\bar{\mathcal{F}}$.

Lemma 9.2. *There exists a constant C so that for each fiber $\bar{F} \in \bar{\mathcal{F}}$ and each infinite normal subgroup $G' \subset G$ there is an element $g \in G' \setminus \{1\}$ so that $d_H(\bar{F}, g(\bar{F})) \le C$.*

Proof. Consider the coarse base B of the coarse fibration $\bar{\mathcal{F}}$. Without loss of generality we will assume that B has bounded geometry and $\bar{F} \in B$; we have a quasi-action $\rho : G \curvearrowright B$. The constant C will depend only on the constants (L, A) of the quasi-action ρ. Since $G \curvearrowright X$ is cocompact, there exists a sequence of distinct elements $g_i \in G$ so that the sequence $\rho(g_i)$ converges to the identity uniformly on compacts in

B. Let $h \in G'$ be an element of infinite order. If $[h, g_i] = 1$ for all $i \geq i_0$, then the elements g_i, $i \geq i_0$, belong to a maximal elementary subgroup $E(h)$ of G containing h. If this is the case we replace h with a hyperbolic element $h' \in G'$ so that the maximal elementary subgroups $E(h), E(h')$ have finite intersection; then $[g_i, h'] \neq 1$ for all but finitely many i's. Passing to a subsequence, we conclude that there exists a hyperbolic element $h \in G'$ and a sequence of distinct elements $g_i \in G$ so that $\rho(g_i)$ converges to the identity uniformly on compacts in B and $[h, g_i] \neq 1$ for all i.

Now pick $R \in \mathbb{R}_+$ so that $\rho(g_i)|B_R(\bar{F}) = Id$ and $R \geq d_H(\bar{F}, h(\bar{F})) + 2A$. Then, since ρ is a quasi-action with constants (L, A),

$$d(\rho([g_i^{-1}, h^{-1}]), \rho(g_i^{-1}) \circ \rho(h^{-1}) \circ \rho(g_i) \circ \rho(h)) \leq 4A,$$

$$d_H(\bar{F}, \rho(g_i^{-1}) \circ \rho(h^{-1}) \circ \rho(g_i) \circ \rho(h)(\bar{F})) \leq 2A,$$

hence

$$d_H(\bar{F}, \rho([g_i^{-1}, h^{-1}])\bar{F}) \leq C := 6A. \quad \square$$

Lemma 9.3. *For each nonelementary hyperbolic group G there exists a sequence of infinite normal subgroups $G_i \subset G$ with trivial intersection.*

Proof. Consider a sequence of hyperbolic elements $g_i \in G$ whose translation length diverges to infinity. Then take G_i to be the normal closure of $\langle g_i \rangle$ in G. $\square$

We will need the following variant of Lemma 8.2:

Lemma 9.4. *There exists a constant D (which depends only on the coarse fibration $\bar{\mathcal{F}}$), so that given any constant $R > 0$ and a fiber $\bar{F} \in \bar{\mathcal{F}}$, there exists an element $g \in G$ and a subset $(x_i)_{i \in \mathbb{Z}} \in \bar{F}$ which is a D-*pseudo-orbit* of g, i.e.,*

$$d(x_{i+1}, g(x_i)) < D$$

for all $i \in \mathbb{Z}$. In addition, the translation length of g is at least R.

Proof. Using Lemma 9.3 we find an infinite normal subgroup $G' \subset G$ such that the translation length of each nontrivial element of G' is at least R. By Lemma 9.2 there exists D (independent of G') and a nontrivial element $g \in G'$ so that $d_H(\bar{F}, g\bar{F}) \leq D$. Pick a point $x_0 \in \bar{F}$. Then there exists $x_1 \in \bar{F}$ so that $d(g(x_0), x_1) \leq D$. The point $g(x_1)$ will lie within distance $\leq D$ of some point $x_2 \in \bar{F}$, etc. In the same fashion we get points x_i for $i < 0$. $\square$

Observe that we can assume that $R > 0$ and hence the element g in the above lemma has infinite order, i.e. is hyperbolic. Let η, ζ be fixed points of g in $\partial_\infty X$; let γ denote the union of all geodesics in X asymptotic to ξ and η. Then γ is a g-invariant (L_0, A_0)-quasi-geodesic with (L_0, A_0) independent of g. The quasi-geodesic γ is called an *axis* of g.

We let $\alpha : \gamma \to \mathbb{R}$ be a quasi-isometry with the coarse inverse $\bar{\alpha}$. Let $f : \mathbb{R} \to X$ denote a uniformly proper map whose image is $\bar{\mathcal{F}}$; let $\bar{f} : \bar{F} \to \mathbb{R}$ be a coarse inverse of f. Let $\pi : X \to \gamma$ denote the nearest-point projection to γ.

We choose $t_i \in \mathbb{R}$ so that $x_i = f(t_i)$, $i \in \mathbb{Z}$. Then g determines an (L, A) quasi-isometry $\psi(g) := \bar{f} \circ g \circ f : \mathbb{R} \to \mathbb{R}$ with (L, A) independent of R. Observe that there exists $D_1 = D_1(D, L, A)$ so that $\{t_i, i \in \mathbb{Z}\}$ is a D_1-pseudo-orbit of $\psi(g)$. The isometry $g : \gamma \to \gamma$ also induces a quasi-isometry $\phi(g) : \mathbb{R} \to \mathbb{R}$ defined by

$$\phi(g) := \alpha \circ g \circ \bar{\alpha}.$$

Lemma 9.5. *Suppose that $h : \mathbb{R} \to \mathbb{R}$ is an (L, A) quasi-isometry which preserves the orientation on $\mathbb{R}$. Then h is coarsely increasing, i.e. there exists a number $C = C(L, A)$ such that if $t - t' \geq C$ then $h(t) - h(t') \geq L^{-1}C - A > 0$.*

Proof. Note that $|h(t) - h(t')| \geq L^{-1}|t - t'| - A$. Hence we only have to show that $h(t) > h(t')$ for an appropriate C. If $h(t) < h(t')$ for some t, t' satisfying $t - t' \geq C$ then $h([t, \infty)) \cap N_A(h((-\infty, t'])) = \emptyset$ implies that $h([t, \infty)) \subset [h(t'), -\infty)$, which contradicts the assumption that h preserves the orientation on $\mathbb{R}$. $\square$

Lemma 9.6. *For each sufficiently large R, $t_{i+1} > t_i$ for all $i \in \mathbb{Z}$.*

Proof. Recall that $|t_{i+1} - \psi(g)(t_i)| \leq D_1$, where D_1 is independent of R. Thus

$$\lim_{R\to\infty} \min_i |t_i - t_{i-1}| = \infty.$$

Now the assertion follows from the previous lemma with $h := \psi(g)$. $\square$

We set $I_j := [t_j, t_{j+1}]$, $j \in \mathbb{Z}$.

Corollary 9.7. *There exists a constant D_2 depending only on the coarse fibration $\bar{\mathcal{F}}$ so that the following holds: The set of intervals $\{I_j, j \in \mathbb{Z}\}$ is a D_2-pseudo-orbit of g, i.e.,*

$$d_H(I_{j+1}, \psi(g)(I_j)) < D_2$$

for all $j \in \mathbb{Z}$.

Proof. We again use Lemma 9.5 with $h := \psi(g)$. For $C := C(L, A)$ we have:

$$t_{j+1} - D_1 \leq h(t_j) < h(t) < h(t_{j+1}) \leq t_{j+2} + D_1$$

for all $j \in \mathbb{Z}$ and all $t \in (t_j + C, t_{j+1} - C)$. Therefore for $D_2 := LC + A + D_1$ we have:

$$h(I_j) \subset N_{D_2}(I_{j+1}).$$

The inclusion $I_{j+1} \subset N_{D_2}(h(I_j))$ is clear as well. $\square$

Set $J_k := \alpha \circ \pi \circ f(I_k), k \in \mathbb{Z}$.

Corollary 9.8. *There exists $D_3 \in \mathbb{R}_+$ independent of R so that the collection of bounded subsets $\{J_k, k \in \mathbb{Z}\}$ of $\mathbb{R}$ forms a D_3-pseudo-orbit of $\phi(g) : \mathbb{R} \to \mathbb{R}$.*

Proof. The assertion follows from the previous corollary since both maps f, π and α are (L, A)-coarse Lipschitz for L and A depending only on the geometry of X and $\bar{\mathcal{F}}$. □

Recall that the displacement of $g : X \to X$ is at least R. We choose R so large that

$$\inf_{y\in\mathbb{R}} \phi(g)(y) - y \geq 2D_3. \tag{9.9}$$

The subset $J_0 \subset \mathbb{R}$ is contained in a finite interval $[T_-, T_+]$. We define $H_+, H_- \subset X$ to be the inverse images of the rays $[T_-, +\infty)$ and $(-\infty, T_+]$ respectively, under the map $\alpha \circ \pi : X \to \mathbb{R}$. Although H_+ is not necessarily quasi-convex in X, there exists $T \leq T_+$ so that H_+ is contained in the quasi-convex hull H of $\alpha \circ \pi^{-1}((-\infty, T])$; moreover

$$\pi(H) \subset (-\infty, T']$$

for some $T' \leq T$, see [9]. In particular, the ideal boundary $\partial_\infty H_+$ of H_+ is a proper subset of $\partial_\infty X$:

$$\partial_\infty H_- \subset \partial_\infty H$$

and the $\partial_\infty H$ does not contain one of the ideal points η, ξ of the quasi-geodesic γ.

Set $\bar{F}_+ := f([t_0, +\infty)), \bar{F}_- := f((-\infty, t_0])$.

Lemma 9.10. *$F_+ \subset H_+$ and hence $\partial_\infty \bar{F}_+$ is a proper subset of $\partial_\infty X = \partial_\infty G$.*

Proof. The subset $J_0 \subset \mathbb{R}$ is contained in an interval $[T_-, T_+]$. Let's verify that for each $i \geq 0$ the interval J_i is also contained in $[T_-, +\infty)$. We argue by induction:

Suppose that $J_k \subset [T_-, +\infty)$. Then, by (9.9),

$$\phi(g)(J_k) \subset [T_- + 2D_3, +\infty),$$

where D_3 is as in Corollary 9.8. Since $d_H(\phi(g)J_k, J_{k+1}) \leq D_3$, we get:

$$J_{k+1} \subset [T_- + 2D_3 - D_3, +\infty) \subset [T_-, +\infty).$$

Thus the union $\cup_{i\geq 0} J_i$ is contained in $[T_-, +\infty)$. It follows that

$$\bar{F}_+ = \cup_{i\geq 0} f(I_i) \subset H_+.$$

Since $\partial_\infty H_+$ is a closed subset of $\partial_\infty X$ disjoint from one of the ideal points of the quasi-geodesic γ the assertion follows. □

Note that the same conclusion $\partial_\infty \bar{F}_- \subset \partial_\infty H_-$ holds for $\partial_\infty \bar{F}_+$ but we do not need this for our argument.

We now can finish the proof of Theorem 9.1. Recall that for each $h \in G$, $h(\bar{F}_+)$ is Hausdorff-close to $\bar{F}_+$ (since G preserves the orientation on the fibers of $\bar{\mathcal{F}}$). Hence

$$h(\partial_\infty \bar{F}_+) = \partial_\infty \bar{F}_+.$$

Therefore the entire group G preserves a proper closed subset of $\partial_\infty G$. This is impossible unless the group G is elementary. □

Question 9.11. Is it true that no nonelementary hyperbolic group admits a coarse fibration by copies of the Euclidean n-space?

References

[1] R. Bieri, W. Neumann, and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math., 90 (1987), pp. 451–477.

[2] B. Bowditch, *Planar groups and the Seifert conjecture*. Preprint, 1999.

[3] D. Calegari, *Bounded cochains on 3-manifolds*. Preprint, 2001.

[4] A. L. Edmonds, *Transformation groups and low-dimensional manifolds*, in "Group actions on manifolds" (Boulder, Colo., 1983), Amer. Math. Soc., 1985, pp. 339–366.

[5] B. Evans and L. Moser, *Solvable fundamental groups of compact 3-manifolds*, Transactions of AMS, 168 (1972), pp. 189–210.

[6] A. Hinkkanen, *Abelian and nondiscrete convergence groups on the circle*, Trans. Amer. Math. Soc., 318 (1990), pp. 87–121.

[7] M. Kapovich and B. Kleiner, *Coarse Alexander duality and duality groups*. To appear in Journal of Differential Geometry.

[8] ——, *Laminations in 3-manifolds*. Preprint, in preparation.

[9] ——, *Geometry of Gromov-hyperbolic spaces satisfying coarse Poincare duality*. Preprint, 2004.

[10] ——, *Geometry of quasi-planes*. Preprint, 2004.

[11] M. Kapovich and B. Leeb, *On asymptotic cones and quasi-isometry classes of fundamental groups of* 3*-manifolds*, GAFA, 5 (1995), pp. 582–603.

[12] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*, Inst. Hautes Études Sci. Publ. Math., 86 (1997), pp. 115–197 (1998).

[13] S. Maillot, *Quasi-isometries, groupes de surfaces et orbifolds fibrs de Seifert*. Ph.D. Thesis.

[14] G. Mess, *The Seifert conjecture and groups which are coarse quasi–isometric to planes*. Preprint, 1990.

[15] P. Scott, *There are no fake Seifert fibre spaces with infinite* π_1, Ann. of Math., 117 (1983), pp. 35–70.

Michael Kapovich:

Department of Mathematics,
1 Shields Ave.,
University of California
Davis, CA 95616
kapovich@math.ucdavis.edu

Bruce Kleiner:

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
bkleiner@math.lsa.umich.edu