ESP
Kouba
Worksheet 6 Solutions

1.) $f(x,y,z)=0$ with $x=u-t$, $y=t$, $z=u$;

$\left(\frac{\partial}{\partial t}\right) \to f_x \cdot x_t + f_y \cdot y_t + f_z \cdot z_t = 0$

$\to f_x \cdot (-1) + f_y \cdot (1) + f_z \cdot (0) = 0$

$\to \boxed{-f_x + f_y = 0}$ and

$\left(\frac{\partial}{\partial u}\right) \to f_x \cdot x_u + f_y \cdot y_u + f_z \cdot z_u = 0$

$\to f_x \cdot (1) + f_y \cdot (0) + f_z \cdot (1) = 0$

$\to \boxed{f_x + f_z = 0}$ so add equations

getting $f_y + f_z = 0$.

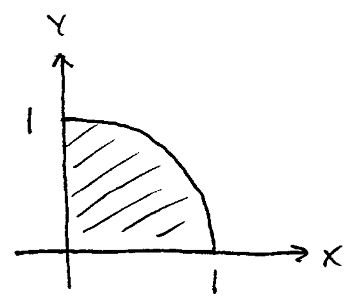
2.) a.) $\int_0^{\frac{\pi}{4}} \int_0^{\cos\theta} 3r^2 \sec\theta \, dr \, d\theta$

$= \int_0^{\frac{\pi}{4}} r^3 \sec\theta \Big|_{r=0}^{r=\cos\theta} d\theta = \int_0^{\frac{\pi}{4}} \cos^3\theta \cdot \sec\theta \, d\theta$

$= \int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1+\cos 2\theta) \, d\theta$

$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2}\right) = \frac{\pi}{8} + \frac{1}{4}$

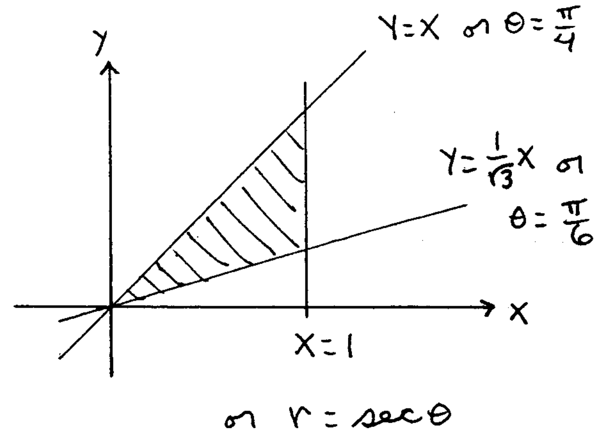
b.) $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2} \, dy \, dx$



$= \int_0^{\frac{\pi}{2}} \int_0^1 e^{r^2} \cdot r \, dr \, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2} e^{r^2} \Big|_0^1 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(e-1) \, d\theta = \frac{\pi}{4}(e-1)$.

c.) $\int_0^1 \int_{\frac{x}{\sqrt{3}}}^{x} \sqrt{x^2+y^2}\, dy\, dx$



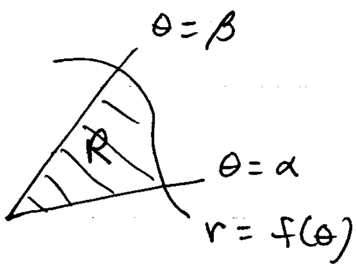
$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_0^{\sec\theta} r \cdot r\, dr\, d\theta$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{3} r^3 \Big|_0^{\sec\theta} d\theta = \frac{1}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^3\theta\, d\theta$

why? $\rightarrow$ $= \frac{1}{3} \cdot \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{1}{6}\left(\sqrt{2} + \ln|\sqrt{2}+1|\right) - \frac{1}{6}\left(\frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right|\right)$

3.)

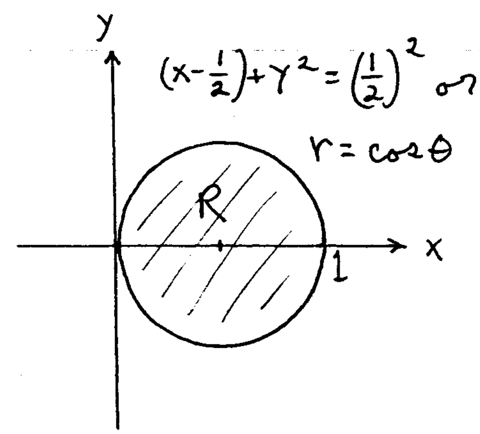


$\text{Area} = \int_R 1\, dA = \int_R 1\, dy\, dx$

$= \int_\alpha^\beta \int_0^{f(\theta)} r\, dr\, d\theta$

$= \int_\alpha^\beta \frac{1}{2} r^2 \Big|_0^{f(\theta)} d\theta = \frac{1}{2} \int_\alpha^\beta [f(\theta)]^2\, d\theta$.

4.) $\text{Vol} = \int_R z\, dA = \int_R (x+1)\, dy\, dx$



$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} (r\cos\theta + 1) \cdot r\, dr\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos\theta} (r^2\cos\theta + r)\, dr\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^3}{3}\cos\theta + \frac{r^2}{2}\right)\Big|_{r=0}^{r=\cos\theta} d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{3}\cos^4\theta + \frac{1}{2}\cos^2\theta\right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3}\left(\frac{1+\cos 2\theta}{2}\right)^2 + \frac{1}{2}\left(\frac{1+\cos 2\theta}{2}\right)\right] d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{12}\left(1+2\cos 2\theta + \cos^2 2\theta\right) + \frac{1}{4}\left(1+\cos 2\theta\right)\right] d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{1}{3} + \frac{5}{12}\cos 2\theta + \frac{1}{12}\left(\frac{1+\cos 4\theta}{2}\right)\right] d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[3/8 + \frac{5}{12}\cos 2\theta + \frac{1}{24}\cos 4\theta\right] d\theta$$

$$= \left[\frac{3}{8}\theta + \frac{5}{24}\sin 2\theta + \frac{1}{96}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{3}{8}\left(\frac{\pi}{2}\right) - \frac{3}{8}\left(-\frac{\pi}{2}\right) = \frac{3}{8}\pi .$$

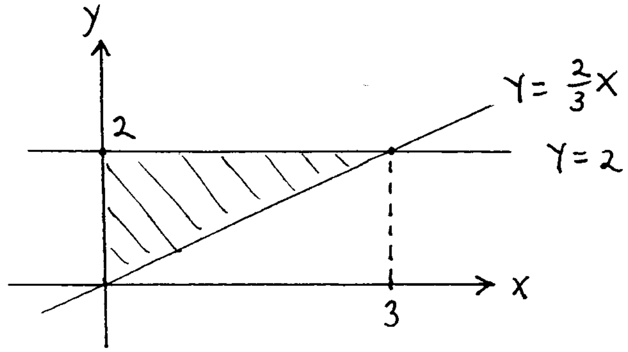
5.) surface : $z = x^2 + y^4$; plane : $z = Ax + By + C$ $\Big\}$ $z_x = 2x$, $z_y = 4y^3$; $z_x = A$, $z_y = B$,

partial derivatives at $(1,-1,2)$ must be equal :

$$\left.\begin{array}{l} 2(1) = A \\ 4(-1)^3 = B \end{array}\right\} \quad \begin{array}{l} A = 2 \\ B = -4 \end{array} \qquad \text{and} \quad 2 = A(1) + B(-1) + C$$

$$\rightarrow 2 = 2(1) - 4(-1) + C \rightarrow C = -4 \rightarrow \boxed{z = 2x - 4y - 4} .$$

6.)



$y = \frac{2}{3}x$ or $x = \frac{3}{2}y$

a.) $$\overline{X} = \frac{\int_0^3 \int_{\frac{2}{3}X}^{2} X \, dy \, dx}{\int_0^3 \int_{\frac{2}{3}X}^{2} 1 \, dy \, dx}$$

$$\overline{Y} = \frac{\int_0^3 \int_{\frac{2}{3}X}^{2} Y \, dy \, dx}{\int_0^3 \int_{\frac{2}{3}X}^{2} 1 \, dy \, dx}$$

b.) $$\overline{X} = \frac{\int_0^3 \int_{\frac{2}{3}X}^{2} X \cdot (X^2+Y) \, dy \, dx}{\int_0^3 \int_{\frac{2}{3}X}^{2} (X^2+Y) \, dy \, dx}$$

$$\overline{Y} = \frac{\int_0^3 \int_{\frac{2}{3}X}^{2} Y \, (X^2+Y) \, dy \, dx}{\int_0^3 \int_{\frac{2}{3}X}^{2} (X^2+Y) \, dy \, dx}$$

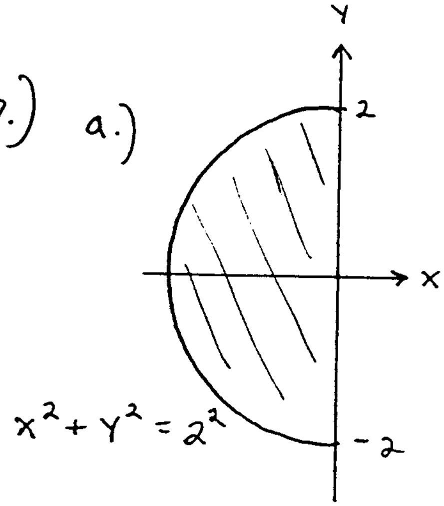
c.) i.) $$M = \int_0^3 \int_{\frac{2}{3}X}^{2} (X-1) \cdot (X^2+Y) \, dy \, dx$$

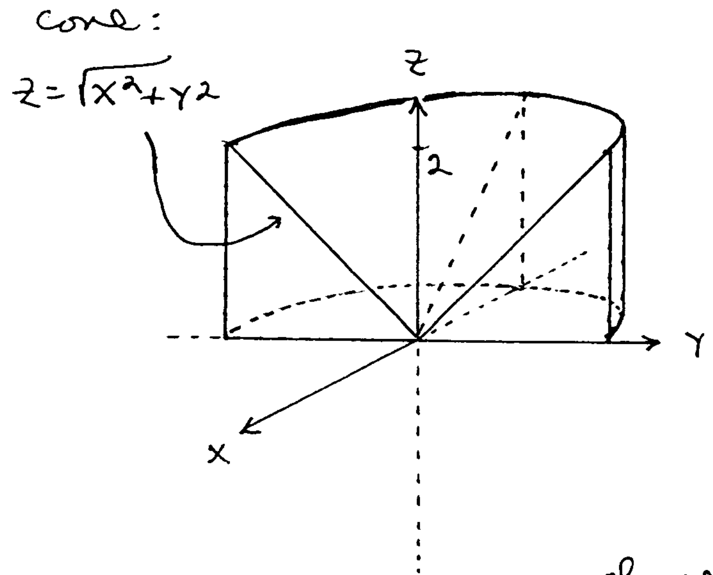
ii.) $$M = \int_0^3 \int_{\frac{2}{3}X}^{2} (Y-2) \, (X^2+Y) \, dy \, dx$$

d.) i.) $$I = \int_0^3 \int_{\frac{2}{3}X}^{2} (X^2+Y^2) \, (X^2+Y) \, dy \, dx$$

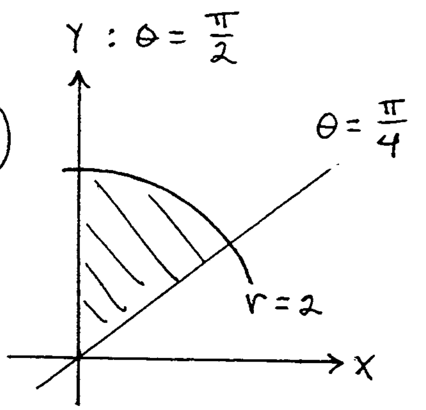
ii.) $$I = \int_0^3 \int_{\frac{2}{3}X}^{2} (X-4)^2 \, (X^2+Y) \, dy \, dx$$

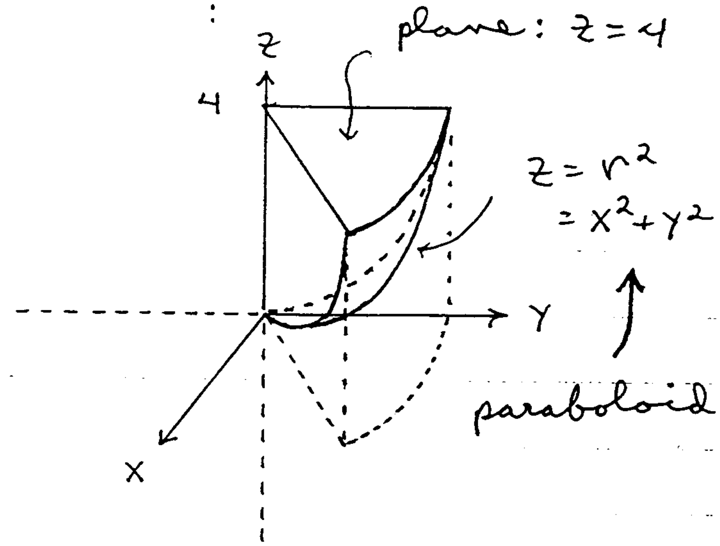
7.) a.)



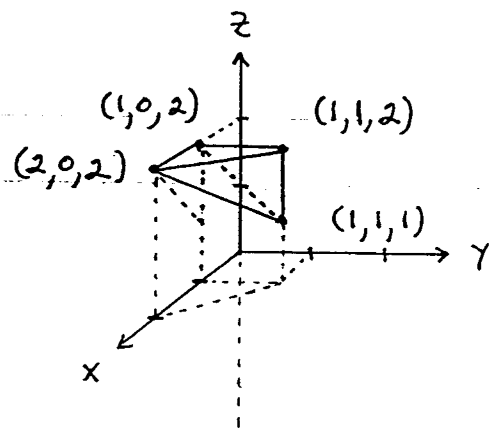


b.)





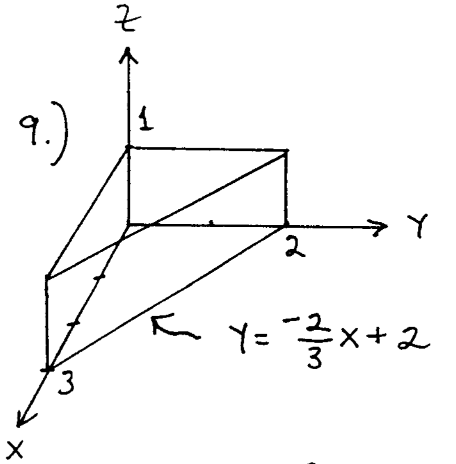
8.)



a.) $1 \le x \le 2$, $0 \le y \le 2-x$,
$2-y \le z \le 2$

b.) $1 \le x \le 2$, $x \le z \le 2$,
$2-z \le y \le 2-x$

9.)



$$\int_R 1 \, dV = \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^1 1 \, dz \, dy \, dx$$

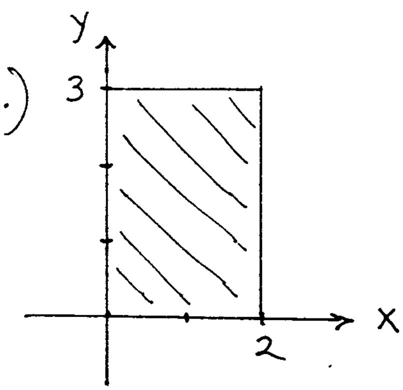
$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} z \Big|_0^1 \, dy \, dx$$

$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} 1 \, dy \, dx$$

$$= \int_0^3 Y \Big|_0^{-\frac{2}{3}x+2} dx = \int_0^3 \left(-\frac{2}{3}x+2\right) dx = \left(-\frac{1}{3}x^2+2x\right)\Big|_0^3 = 3,$$

the volume of the prism.

10.)



$$\int_R z \, dV = \int_0^2 \int_0^3 \int_0^{x+2Y} z \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^3 \frac{1}{2} z^2 \Big|_0^{x+2Y} dz \, dy \, dx$$

$$= \int_0^2 \int_0^3 \frac{1}{2}(x+2Y)^2 dy \, dx = \frac{1}{2} \int_0^2 \int_0^3 (x^2+4xY+4Y^2) \, dy \, dx$$

$$= \frac{1}{2} \int_0^2 \left(x^2Y + 2xY^2 + \frac{4}{3}Y^3\right)\Big|_{Y=0}^{Y=3} dx$$

$$= \frac{1}{2} \int_0^2 (3x^2 + 18x + 36) \, dx$$

$$= \frac{1}{2} (x^3 + 9x^2 + 36x) \Big|_0^2$$

$$= 58$$