ESP
Kouba
Worksheet 8 Solutions

1.) a.) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \cdots$ and $\left(\frac{1}{2}\right)^n = \frac{1}{2^n} \to 0$

b.) $.9999, .99980001, .99970003, .99960006, .999501, \cdots$
and $(.9999)^n \to 0$

c.) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots$ and $\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to \frac{1}{1+0} = 1$

d.) $2, 2.295225, 2.37037037, 2.44140625, 2.48832, \cdots$
and $\left(1+\frac{1}{n}\right)^n \to e$.

e.) $3, 4, 4.62962963, 5.0625, 5.37824, \cdots$ and
$$\left(1+\frac{2}{n}\right)^n = \left(1+\frac{1}{\left(\frac{n}{2}\right)}\right)^{\left(\frac{n}{2}\right)\cdot 2} \to e^2$$

f.) $3, 6.25, 12.7037037, 25.62890625, 51.53632, \cdots$
and $2^n < \left(2+\frac{1}{n}\right)^n$ and since $2^n \to \infty$,
it follows that $\left(2+\frac{1}{n}\right)^n \to \infty$.

g.) $4, 5.0625, 5.618655693, 5.960464477, 6.191736422, \cdots$
and $\left(1+\frac{1}{n}\right)^{2n} = \left(1+\frac{1}{n}\right)^{n\cdot 2} \to e^2$

h.) $.5, .444444444, .421875, .4096, .401877572, \cdots$
and $\left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e}$

i.) $-.841470984, -.958851077, -.98158409, -.989615837,$
$-.993346653, \cdots$ and
$$n\cdot\cos\left(\frac{\pi}{2}+\frac{1}{n}\right) = \frac{\cos\left(\frac{\pi}{2}+\frac{1}{n}\right)}{\frac{1}{n}} \overset{\text{"}\frac{0}{0}\text{"}}{\longrightarrow} \frac{-\sin\left(\frac{\pi}{2}+\frac{1}{n}\right)\cdot\cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} \to -1$$

j.) $\frac{2}{3}, \frac{6}{11}, \frac{12}{25}, \frac{20}{43}, \frac{30}{71}, \cdots$ and

$$\frac{n^2+n}{3n^2-n+1} = \frac{1+1/n}{3-\frac{1}{n}+\frac{1}{n^2}} \to \frac{1}{3}$$

k.) $2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}, \cdots$ and $\frac{2^n}{n!} \to 0$ (SEE book)

l.) $50, 1250, 20833.3, 260416.7, 2604166.7, \ldots$ and $\frac{50^n}{n!} \to 0$ (SEE book)

m.) $3, 1.732050808, 1.44224957, 1.316074013, 1.24573094, \cdots$ and $3^{1/n} \to 3^0 = 1$

n.) $1, 1.414213562, 1.44224957, 1.414213562, 1.379729661, \cdots$ and

$$\ln n^{1/n} = \frac{1}{n}\ln n = \frac{\ln n}{n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\longrightarrow} \frac{\frac{1}{n}}{1} \to 0 \text{ so}$$

$n^{1/n} \to 1$

o.) $0, .34657359, .366204096, .34657359, .321887582, \cdots$ and

$$\frac{\ln n}{n} \overset{\frac{\infty}{\infty}}{\longrightarrow} \frac{\frac{1}{n}}{1} = 0$$

p.) $0, .832554611, 1.03184584, 1.08508526, 1.099853511, \cdots$ and

$$\ln(\ln n)^{1/n} = \frac{1}{n}\ln(\ln n) = \frac{\ln(\ln n)}{n} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\longrightarrow} \frac{\frac{1}{\ln n}\cdot\frac{1}{n}}{1} \to 0$$

so $(\ln n)^{1/n} \to 1$

q.) $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{4}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}, \cdots$ and

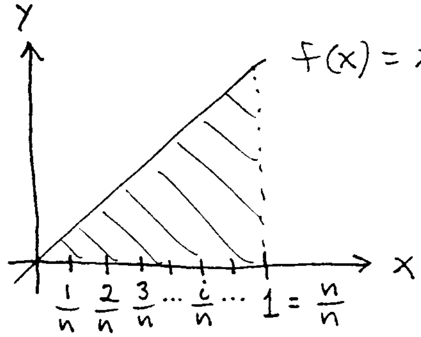
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i-1} = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{i-1} = 1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots = \frac{1}{1-\frac{1}{2}} = 2$$

r.) $1, 1+2, 1+2+3, 1+2+3+4, 1+2+3+4+5, \cdots$ and

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{1}{2}(n^2+n) \longrightarrow \infty \text{ (diverges)}$$

s.) $1, \frac{3}{4}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}, \cdots$ and $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$ OR

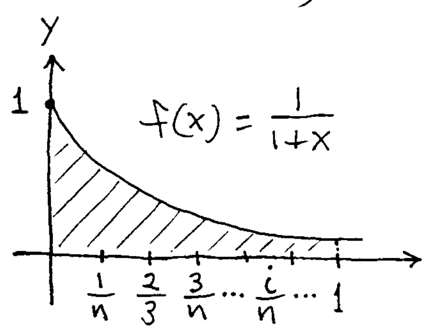


let $x_i = \frac{i}{n}$ so that

$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)\cdot\frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$

$= \int_0^1 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$.

t.) $\frac{1}{2}, \frac{1}{3}+\frac{1}{4}, \frac{1}{4}+\frac{1}{5}+\frac{1}{6}, \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}, \frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}, \cdots$ and



let $x_i = \frac{i}{n}$ so that

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{1+\left(\frac{i}{n}\right)} \cdot \frac{1}{n}$

$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i = \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x}\,dx = \ln(1+x)\Big|_0^1 = \ln 2$

2.) a.) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = \frac{1}{1-\frac{1}{2}} = 2$

b.) $\sum_{n=1}^{\infty} (.9999)^n = (.9999) + (.9999)^2 + (.9999)^3 + \cdots$

$= (.9999)\cdot\left[1 + (.9999) + (.9999)^2 + (.9999)^3 + \cdots\right]$

$= (.9999) \cdot \frac{1}{1-(.9999)} = 9999$

c.) $\sum_{n=1}^{\infty} 3\left(\frac{-1}{2}\right)^n = 3\cdot\left(\frac{-1}{2}\right) + 3\left(\frac{-1}{2}\right)^2 + 3\left(\frac{-1}{2}\right)^3 + \cdots$

$$= 3\left(\frac{-1}{2}\right)\cdot\left[1+\left(\frac{-1}{2}\right)+\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{2}\right)^3+\cdots\right] = \frac{-3}{2}\cdot\frac{1}{1-\left(\frac{-1}{2}\right)} = -1$$

d.) $$\sum_{n=8}^{\infty}\left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^7+\left(\frac{1}{2}\right)^8+\left(\frac{1}{2}\right)^9+\cdots$$

$$= \left(\frac{1}{2}\right)^7\cdot\left[1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2+\cdots\right] = \left(\frac{1}{2}\right)^7\cdot\frac{1}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

e.) $$\sum_{n=1}^{\infty}4^{-n} = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \left(\frac{1}{4}\right)\cdot\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\cdots\right] = \frac{1}{4}\cdot\frac{1}{1-\frac{1}{4}} = \frac{1}{3}$$

f.) $\sum_{n=1}^{\infty}(-4)^n$ diverges since $r=-4<-1$

g.) $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right) = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots = \infty$ (diverges)

h.) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ diverges since

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \neq 0$ (divergence test)

i.) $\sum_{n=1}^{\infty}\frac{n}{100n+1}$ diverges since

$\lim_{n\to\infty}\frac{n}{100n+1} = \lim_{n\to\infty}\frac{1}{100+\frac{1}{n}} = \frac{1}{100} \neq 0$ (divergence test)

j.) sequence of partial sums:

$s_1 = \frac{1}{2}-\frac{1}{3}$

$s_2 = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{2}-\frac{1}{4}$

$s_3 = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right) = \frac{1}{2}-\frac{1}{5}$

$s_4 = \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right) = \frac{1}{2}-\frac{1}{6}$

$\vdots$

$s_n = \frac{1}{2}-\frac{1}{n+2}$ so that

$$\sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right] = \lim_{n\to\infty}s_n = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right) = \frac{1}{2}.$$

k.) $\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \sum_{n=1}^{\infty} \left[\frac{2}{n} + \frac{-2}{n+1} \right] = 2 \sum_{n=1}^{\infty} \left[\frac{1}{n} + \frac{-1}{n+1} \right]$ so sequence of partial sums is :

$S_1 = 2\left(1-\frac{1}{2}\right)$

$S_2 = 2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)\right] = 2\left(1-\frac{1}{3}\right)$

$S_3 = 2\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)\right] = 2\left(1-\frac{1}{4}\right)$

$\vdots$

$S_n = 2\left(1-\frac{1}{n+1}\right)$, then

$$\sum_{n=1}^{\infty} \frac{2}{n^2+n} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} 2\left(1-\frac{1}{n+1}\right) = 2.$$

l.) $1+\left(\frac{-1}{2}\right)+\left(\frac{-1}{2}\right)^2+\left(\frac{-1}{2}\right)^3+\cdots = \frac{1}{1-\left(\frac{-1}{2}\right)} = \frac{2}{3}$

m.) $1+\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

n.) $3-2+\frac{4}{3}-\frac{8}{9}+\frac{16}{27}-\frac{32}{81}+\cdots = 3\left[1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\frac{16}{81}-\cdots\right]$

$= 3\cdot\left[1+\left(\frac{-2}{3}\right)+\left(\frac{-2}{3}\right)^2+\left(\frac{-2}{3}\right)^3+\cdots\right] = 3\cdot\frac{1}{1-\left(\frac{-2}{3}\right)} = \frac{9}{5}$

o.) Let $f(x)=x^2$ and $x_i = 3+\frac{i}{n}$ on $[3,4]$ then

$\lim_{n\to\infty} \sum_{i=1}^{n} \left(3+\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$

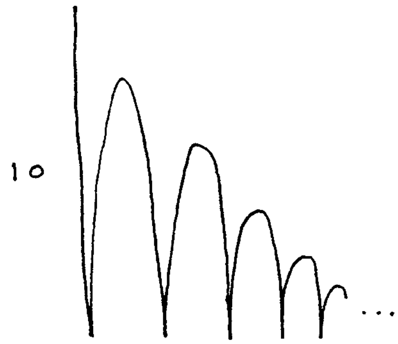
$= \int_3^4 f(x)\,dx = \int_3^4 x^2\,dx = \frac{37}{3}$

p.) Let $f(x)=\frac{5}{x}$ and $x_i = 1+\frac{2i}{n}$ on $[1,3]$ then

$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{5}{1+\frac{2i}{n}}\right)\cdot\frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot \Delta x_i$

$= \int_1^3 f(x)\,dx = \int_1^3 \frac{5}{x}\,dx = 5\cdot \ln x \Big|_1^3 = 5\cdot \ln 3$

3.)



$$\text{Total} = 10 + 2(10)(.75) + 2(10)(.75)^2 + 2(10)(.75)^3 + \cdots$$
$$= 10 + 2(10)(.75)\cdot\left[1 + (.75) + (.75)^2 + (.75)^3 + \cdots\right]$$
$$= 10 + 15 \cdot \frac{1}{1-(.75)} = 70 \text{ ft.}$$

4.) Limit is

$$3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cdots}}}} = x \longrightarrow 3 + \frac{1}{x} = x \longrightarrow$$

$$0 = x^2 - 3x - 1 \rightarrow x = \frac{3 \pm \sqrt{9+4}}{2}$$

$$\rightarrow x = \frac{3 + \sqrt{13}}{2}$$