Math 16B
Kouba
Differentiating the Natural Logarithm

FACT : $\lim_{k \to 0} (1+k)^{1/k} = e \approx 2.71828$

Let $f(x) = \ln x$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h \to 0} \ln\left(\frac{x}{x} + \frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h \to 0} \ln\left[\left(1 + \frac{h}{x}\right)^{\frac{1}{h/x}}\right]^{\frac{1}{x}}$$

$$= \ln\left[e\right]^{\frac{1}{x}}$$

$$= \frac{1}{x} \, .$$

RULE : $D\{\ln x\} = \frac{1}{x}$

CHAIN RULE : $D\{\ln f(x)\} = \frac{1}{f(x)} \cdot f'(x)$