Math 16B
Section 4.4

Natural Logarithms

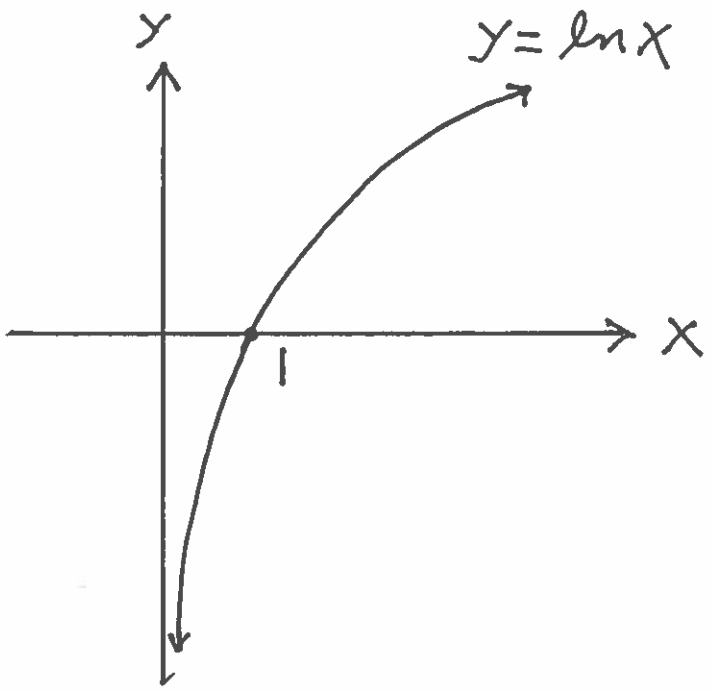
Definition: The *natural logarithm* of x equals b, written

$$\ln x = b,$$

is equivalent to the exponential equation

$$e^b = x.$$

Example: Sketch the graph of $y = \ln x$.



NOTE: $\ln x$ is not defined for $x = 0$ and $x < 0$.

NOTE: y is a one-to-one function (passes vertical and horizontal line tests), so it has an inverse function:

$y = \ln x \longrightarrow$ (switch variables)

$x = \ln y \longrightarrow$ (solve for y)

$y = e^x$; thus

$y = e^x$ and $y = \ln x$ are inverses.

Rules for Logarithms

1.) $\ln 1 = 0$ (since $e^0 = 1$)

2.) $\ln e = 1$ (since $e^1 = e$)

3.) $\ln(e^x) = x$

4.) $e^{\ln x} = x$

5.) $\ln(x \cdot y) = \ln x + \ln y$

6.) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

7.) $\ln x^m = m \ln x$

Why is 5.) true?

assume $\ln x = m$ and $\ln y = n \rightarrow$

$x = e^m$ and $y = e^n$. Then

$$\ln(xy) = \ln(e^m \cdot e^n)$$

$$= \ln(e^{m+n})$$

$$= m + n$$

$$= \ln x + \ln y$$

Please Avoid These Common Mistakes

1.) $\ln(x \cdot y) = \ln x \cdot \ln y$

2.) $\ln\left(\frac{x}{y}\right) = \frac{\ln x}{\ln y}$

3.) a.) $\ln(x+y) = \ln x + \ln y$

b.) $\ln(x-y) = \ln x - \ln y$

4.) $(\ln x)^m = m \cdot \ln x$

5.) $\ln(e^m + e^n) = m + n$

Why is 4.) not true?

Let $x=e$ and $m=2$:

$(\ln x)^m = (\ln e)^2 = (1)^2 = 1$, but

$m \cdot \ln x = 2 \cdot \ln e = 2 \cdot (1) = 2$, so

$(\ln x)^m \neq m \cdot \ln x$

Example: Solve for t.

1.) $2e^{-t} = 3 \rightarrow e^{-t} = \frac{3}{2} \rightarrow$

$\ln e^{-t} = \ln\left(\frac{3}{2}\right) \rightarrow -t = \ln\left(\frac{3}{2}\right) \rightarrow$

$$\boxed{t = -\ln\left(\frac{3}{2}\right)}$$

2.) $\frac{3}{1+e^{2t}} = \frac{2}{3} \rightarrow 9 = 2 + 2e^{2t} \rightarrow$

$7 = 2e^{2t} \rightarrow \frac{7}{2} = e^{2t} \rightarrow$

$\ln\left(\frac{7}{2}\right) = \ln\left(e^{2t}\right) = 2t \rightarrow$

$$\boxed{t = \frac{1}{2}\ln\left(\frac{7}{2}\right)}$$

3.) $(\ln t)^2 - 2\ln t = 0 \rightarrow$

$\ln t(\ln t - 2) = 0 \rightarrow$

$\ln t = 0 \rightarrow t = 1$ or $\ln t = 2 \rightarrow t = e^2$

4.) $e^{2t} - e^t = 6 \rightarrow$

$e^{2t} - e^t - 6 = 0 \rightarrow$

$(e^t)^2 - (e^t) - 6 = 0 \rightarrow$

$(e^t - 3)(e^t + 2) = 0 \rightarrow$

$e^t = 3 \rightarrow \ln e^t = \ln 3 \rightarrow \boxed{t = \ln 3}$ or $e^t = -2$ (Impossible)

5.) $\ln(t+1) + \ln(t+2) = \ln 6 \rightarrow$

$\ln(t+1)(t+2) = \ln 6 \rightarrow$

$\ln(t^2 + 3t + 2) = \ln 6 \rightarrow$

$$e^{\ln(t^2+3t+2)} = e^{\ln 6} \rightarrow$$

$$t^2+3t+2 = 6 \rightarrow$$

$$t^2+3t-4 = 0 \rightarrow$$

$$(t-1)(t+4) = 0 \rightarrow$$

$\boxed{t=1}$ or $t=-4$ (No!)

since $\ln(t+1) = \ln(-3)$!!!

6.) $e^{2t} = 6e^t - 8 \rightarrow$

$$e^{2t} - 6e^t + 8 = 0 \rightarrow$$

$$(e^t)^2 - 6(e^t) + 8 = 0 \rightarrow$$

$$(e^t - 4)(e^t - 2) = 0 \rightarrow$$

$e^t = 4 \rightarrow$ or $e^t = 2 \rightarrow$

$\ln e^t = \ln 4 \rightarrow$ or $\ln e^t = \ln 2 \rightarrow$

$\boxed{t = \ln 4}$ or $\boxed{t = \ln 2}$

Example: A deposit of $5000 is made in a savings account earning an annual interest rate of 3.5% compounded continuously. How long will it take for the account to grow to $10,000?

Begin with $A = Pe^{rt} \rightarrow$

$$10{,}000 = 5000\, e^{0.035t} \rightarrow$$

$$2 = e^{0.035t} \rightarrow$$

$$\ln 2 = \ln e^{0.035t} = 0.035t \rightarrow$$

$$t = \frac{\ln 2}{0.035} \approx 19.8 \text{ yrs.}$$