Math 16B
Section 4.5

Differentiating Logarithmic Functions

What is the derivative of the function

$$f(x) = \ln x \quad ?$$

SEE the next page.

Math 16B
Kouba
Differentiating the Natural Logarithm

<u>FACT</u> : $\lim_{k\to 0}(1+k)^{1/k} = e \approx 2.71828$

Let $f(x) = \ln x$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\ln(x+h)-\ln(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\ln\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \ln\left(\frac{x}{x}+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0} \ln\left[\left(1+\frac{h}{x}\right)^{\frac{1}{h/x}}\right]^{\frac{1}{x}}$$

$$= \ln\left[e\right]^{\frac{1}{x}}$$

$$= \frac{1}{x}\ .$$

<u>RULE</u> : $D\{\ln x\} = \dfrac{1}{x}$

<u>CHAIN RULE</u> : $D\{\ln f(x)\} = \dfrac{1}{f(x)} \cdot f'(x)$

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<u>Example</u>: Differentiate each of the following functions.

1.) $y = x^4 \ln x \xrightarrow{D}$

$y' = x^4 \cdot \frac{1}{x} + 4x^3 \cdot \ln x$

2.) $f(x) = (3x - 2\ln x)^{1/2} \xrightarrow{D}$

$f'(x) = \frac{1}{2}(3x - 2\ln x)^{-1/2} \cdot \left(3 - 2 \cdot \frac{1}{x}\right)$

3.) $y = \ln(x^3 + x^2 + e^x) \xrightarrow{D}$

$y' = \frac{1}{x^3 + x^2 + e^x} \cdot (3x^2 + 2x + e^x)$

4.) $f(x) = \ln(\sin x) \xrightarrow{D}$

$f'(x) = \frac{1}{\sin x} \cdot \cos x$

5.) $y = \tan(\ln x) \xrightarrow{D}$

$y' = \sec^2(\ln x) \cdot \frac{1}{x}$

6.) $y = \ln(\ln(\ln(x^2+1)))$ $\xrightarrow{D}$

$$y' = \frac{1}{\ln(\ln(x^2+1))} \cdot \frac{1}{\ln(x^2+1)} \cdot \frac{1}{x^2+1} \cdot 2x$$

<u>Example</u>: Consider the function $y = x(\ln x)^2$. Solve each of $Y' = 0$ and $Y'' = 0$ for x, and set up a Sign Chart for each.

$\xrightarrow{D}$ $y' = x \cdot 2(\ln x) \cdot \frac{1}{x} + (1)(\ln x)^2$

$= 2\ln x + (\ln x)^2$

$= \ln x\,(2 + \ln x) = 0 \longrightarrow$

$\ln x = 0 \rightarrow x = 1$ or

$\ln x = -2 \rightarrow x = e^{-2}$

NO		+	0	−	0	+	y'
	$x=0$		$x=e^{-2}$		$x=1$		

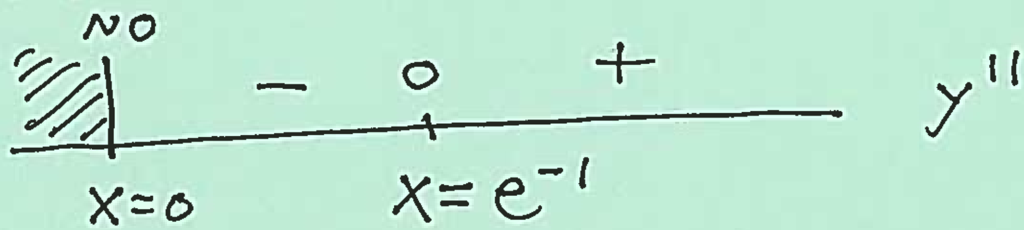
$\xrightarrow{D}$ $Y'' = \ln x \cdot \left(\frac{1}{x}\right) + \frac{1}{x}(2 + \ln x)$

$= \frac{1}{x}(\ln x + 2 + \ln x)$

$$= \frac{1}{x}(2 + 2\ln x)$$

$$= \frac{2}{x}(1 + \ln x) = 0 \longrightarrow$$

$\ln x = -1 \longrightarrow x = e^{-1}$ and $x \neq 0$



<u>Example</u>: Find an equation of the line tangent to the graph of $y = \dfrac{\ln x}{3 + \ln x}$ at $x = e$.

If $x = e$, then $y = \dfrac{\ln e}{3 + \ln e} = \dfrac{1}{3+1} = \dfrac{1}{4}$;

$$\xrightarrow{D} y' = \frac{(3 + \ln x)\cdot \frac{1}{x} - \ln x \cdot \left(\frac{1}{x}\right)}{(3 + \ln x)^2},$$

and $x = e$, so SLOPE is

$$m = y' = \frac{(3 + \ln e)\cdot \frac{1}{e} - \ln e \cdot \left(\frac{1}{e}\right)}{(3 + \ln e)^2}$$

$$= \frac{(3+1)\frac{1}{e} - (1)\left(\frac{1}{e}\right)}{(3+1)^2}$$

$= \frac{1}{16}\left(\frac{4}{e} - \frac{1}{e}\right) = \frac{3}{16e}$; then tangent line is

$$y - \frac{1}{4} = \frac{3}{16e}(x - e) .$$

<u>Question</u> : Why is $D(e^x) = e^x$?

<u>answer</u> : Let $y = e^x \rightarrow$

$$\ln y = \ln e^x = x \xrightarrow{D}$$

$$\frac{1}{y} y' = 1 \rightarrow$$

$$y' = y \rightarrow$$

$$y' = e^x .$$

New Logarithmic and Exponential Derivative Formulas

RECALL: I.) a.) $D(e^x) = e^x$

b.) $D\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$

II.) a.) $D(\ln x) = \frac{1}{x}$

b.) $D(\ln f(x)) = \frac{1}{f(x)} \cdot f'(x)$

NEW RULES:

III.) a.) $D(a^x) = a^x \ln a$

b.) $D\left(a^{f(x)}\right) = a^{f(x)} \cdot f'(x) \cdot \ln a$

IV.) a.) $D(\log_b x) = \frac{1}{x} \cdot \frac{1}{\ln b}$

b.) $D(\log_b f(x)) = \frac{1}{f(x)} \cdot f'(x) \cdot \frac{1}{\ln b}$

PROOF: III.) a.) $y = a^x \longrightarrow$

$\ln y = \ln a^x \longrightarrow$

$$\ln y = x \cdot \ln a \xrightarrow{D}$$

$$\frac{1}{y} y' = \ln a \longrightarrow$$

$$y' = y \cdot \ln a \longrightarrow$$

$$y' = a^x \cdot \ln a \quad .$$

<u>IV.</u>) a.) <u>FACT</u>: $\log_b x = \dfrac{\ln x}{\ln b}$; then

$$y = \log_b x = \frac{1}{\ln b} \ln x \xrightarrow{D}$$

$$y' = \frac{1}{\ln b} \cdot \frac{1}{x} \quad .$$

<u>Example</u>: Differentiate each of the following functions.

1.) $y = x^3 + 3^x \xrightarrow{D}$

$$y' = 3x^2 + 3^x \cdot \ln 3$$

2.) $y = \ln x + \log_{10} x - \log_7 x \xrightarrow{D}$

$$y' = \frac{1}{x} + \frac{1}{x} \cdot \frac{1}{\ln 10} - \frac{1}{x} \cdot \frac{1}{\ln 7}$$

3.) $f(x) = 3^{x-x^2} + \log_5 (x^3+1) \xrightarrow{D}$

$$f'(x) = 3^{x-x^2} \cdot (1-2x) \cdot \ln 3$$
$$+ \frac{1}{x^3+1} \cdot 3x^2 \cdot \frac{1}{\ln 5}$$

4.) $y = x\,2^{4-x} \xrightarrow{D}$

$$y^2 = x \cdot 2^{4-x} \cdot (-1) \ln 2 + (1) 2^{4-x}$$

5.) $f(x) = \dfrac{3x+5}{\log_2 x} \xrightarrow{D}$

$$f'(x) = \frac{\log_2 x \cdot (3) - (3x+5) \cdot \frac{1}{x} \cdot \frac{1}{\ln 2}}{(\log_2 x)^2}$$