Math 16B
Section 5.2

Finding antiderivatives using U-Substitution

RECALL: (Chain Rule)

$$D\, f(g(x)) = f'(g(x)) \cdot g'(x)$$

It follows that

$$\int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

Think of U-Substitution as a method for "reversing" the Chain Rule. Here is how it works:

For $\int f'(g(x))\, g'(x)\, dx$

(Let $u = g(x) \xrightarrow{D} \frac{du}{dx} = g'(x) \longrightarrow$ (??)
$du = g'(x)\, dx$), then

$$\int f'(g(x))\,g'(x)\,dx = \int f'(u)\,du$$

$$= f(u) + c$$

$$= f(g(x)) + c \;.$$

Example: Use u-Substitution to find the following antiderivatives.

1.) $\int 2x\,(x^2+1)^5\,dx$

(Let $u = x^2+1 \xrightarrow{D} du = 2x\,dx$)

$$= \int u^5\,du = \frac{1}{6}u^6 + c = \frac{1}{6}(x^2+1)^6 + c$$

2.) $\int \sqrt{3x+5}\,dx$

(Let $u = 3x+5 \xrightarrow{D} du = 3\,dx$

$\rightarrow \frac{1}{3}\,du = dx$)

$$= \int \sqrt{u}\cdot\frac{1}{3}\,du = \frac{1}{3}\int u^{1/2}\,du$$

$$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + c = \frac{2}{9}(3x+5)^{3/2} + c$$

3.) $\int \frac{7}{(3-x)^2} dx$

(Let $u = 3-x \xrightarrow{D} du = -dx \rightarrow -du = dx$)

$= \int \frac{7}{u^2} \cdot -du = -7 \int u^{-2} du$

$= -7 \cdot -u^{-1} + c = 7(3-x)^{-1} + c$

4.) $\int \frac{x+1}{(x^2+2x)^3} dx$

(Let $u = x^2+2x \xrightarrow{D} du = (2x+2)\,dx$

$\rightarrow du = 2(x+1)\,dx \rightarrow \frac{1}{2} du = (x+1)\,dx$)

$= \int \frac{1}{u^3} \cdot \frac{1}{2} du = \frac{1}{2} \int u^{-3} du$

$= \frac{1}{2} \cdot \frac{-1}{2} u^{-2} + c = \frac{-1}{4}(x^2+2x)^{-2} + c$

5.) $\int \frac{(1+1/t)^4}{t^2} dt$

(Let $u = 1 + \frac{1}{t} = 1 + t^{-1} \xrightarrow{D} du = -t^{-2} dt$

$\rightarrow -du = \frac{1}{t^2} dt$)

$$= \int u^4 \cdot -du = -\frac{1}{5}u^5 + C = \frac{-1}{5}\left(1+\frac{1}{t}\right)^5 + C$$

6.) $\int (x\cos x + \sin x)(x \sin x)^3 \, dx$

(Let $u = x \sin x \to du = (x \cos x + \sin x)\,dx$)

$$= \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}(x \sin x)^4 + C$$

7.) $\int \frac{\sqrt{4+\sqrt{x+4}}}{\sqrt{x+4}}\, dx$

(Let $u = 4 + \sqrt{x+4} \xrightarrow{D} du = \frac{1}{2}(x+4)^{-1/2}\,dx$

$\longrightarrow 2\,du = \frac{1}{\sqrt{x+4}}\,dx$)

$$= \int \sqrt{u} \cdot 2\,du = 2\int u^{1/2}\,du$$

$$= 2 \cdot \frac{2}{3} u^{3/2} + C$$

8.) $\int \frac{\sqrt{4+\ln x}}{x}\, dx$

(Let $u = 4 + \ln x \xrightarrow{D} du = \frac{1}{x}\,dx$)

$$= \int \sqrt{u}\, du = \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{3}(4 + \ln x)^{3/2} + C$$

9.) $\int x(e^{x^2}+1)(e^{x^2}+x^2)^2\, dx$

(Let $u = e^{x^2} + x^2 \xrightarrow{D}$

$du = (2xe^{x^2} + 2x)\, dx = 2x(e^{x^2}+1)\, dx$

$\longrightarrow \frac{1}{2} du = x(e^{x^2}+1)\, dx$)

$$= \frac{1}{2}\int u^2\, du = \frac{1}{2}\cdot\frac{1}{3} u^3 + C$$

$$= \frac{1}{6}(e^{x^2} + x^2)^3 + C$$