Math 16B
Section 5.4

## Definite Integrals

Definition: If $F(x)$ is any antiderivative for $f(x)$, then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

We call $\int_a^b f(x)\,dx$ a definite integral.

Example: Evaluate each Definite Integral.

1.) $\int_1^3 (2x+1)\,dx = (x^2+x)\Big|_1^3$

$= (3^2+3) - (1^2+1) = 12 - 2 = 10$

2.) $\int_{-1}^0 (x^3 - x^2)\,dx = \left(\frac{1}{4}x^4 - \frac{1}{3}x^3\right)\Big|_{-1}^0$

$= \left(\frac{1}{4}(0)^4 - \frac{1}{3}(0)^3\right) - \left(\frac{1}{4}(-1)^4 - \frac{1}{3}(-1)\right)$

$= (0) - \left(\frac{1}{4} + \frac{1}{3}\right) = -\left(\frac{3}{12} + \frac{4}{12}\right) = \frac{-7}{12}$

3.) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x \, dx = \sin x \Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= \sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{6}\right) = 1 - \frac{1}{2} = \frac{1}{2}$

4.) $\int_{0}^{\ln 3} (e^x - 1) \, dx = (e^x - x) \Big|_{0}^{\ln 3}$

$= (e^{\ln 3} - \ln 3) - (e^0 - 0)$

$= 3 - \ln 3 - 1 = 2 - \ln 3$

5.) $\int_{-2}^{2} (3x - 2x^3) \, dx = \left(\frac{3}{2}x^2 - \frac{1}{2}x^4\right) \Big|_{-2}^{2}$

$= \left(\frac{3}{2}(2)^2 - \frac{1}{2}(2)^4\right) - \left(\frac{3}{2}(-2)^2 - \frac{1}{2}(-2)^4\right)$

$= (6 - 8) - (6 - 8) = 0$

6.) $\int_{e-1}^{e^2-1} \frac{1}{x+1} \, dx = \ln|x+1| \Big|_{e-1}^{e^2-1}$

$= \ln|(e^2-1)+1| - \ln|(e-1)+1|$

$= \ln e^2 - \ln e = 2 - 1 = 1$

7.) $\int_0^{\frac{\pi}{9}} \sec^2 3x \, dx = \frac{1}{3} \tan 3x \Big|_0^{\frac{\pi}{9}}$

$= \frac{1}{3} \tan 3\left(\frac{\pi}{9}\right) - \frac{1}{3} \tan 3(0)$

$= \frac{1}{3} \tan\left(\frac{\pi}{3}\right) - \frac{1}{3} \tan(0)$

$= \frac{1}{3}(\sqrt{3}) - \frac{1}{3}(0) = \frac{\sqrt{3}}{3}$

8.) $\int_0^1 x(x^2+2)^3 \, dx$

(Let $u = x^2 + 2 \xrightarrow{D} du = 2x\,dx \longrightarrow$
$\frac{1}{2} du = x\,dx$ ; and $x: 0 \to 1$, so
$u: (0)^2 + 2 \longrightarrow (1)^2 + 2$, i.e., $u: 2 \to 3$)

$= \frac{1}{2} \int_2^3 u^3 \, du = \frac{1}{2} \cdot \frac{1}{4} u^4 \Big|_2^3$

$= \frac{1}{8}(3)^4 - \frac{1}{8}(2)^4 = \frac{81}{8} - \frac{16}{8} = \frac{65}{8}$

9.) $\int_0^{\ln 2} \frac{e^x}{e^x+1} \, dx$

(Let $u = e^x + 1 \xrightarrow{D} du = e^x \, dx$)

$$= \int_{x=0}^{x=\ln 2} \frac{1}{u}\, du = \ln|u| \Big|_{x=0}^{x=\ln 2}$$

$$= \ln|e^x+1| \Big|_0^{\ln 2}$$

$$= \ln|e^{\ln 2}+1| - \ln|e^0+1|$$

$$= \ln(2+1) - \ln(1+1)$$

$$= \ln 3 - \ln 2$$

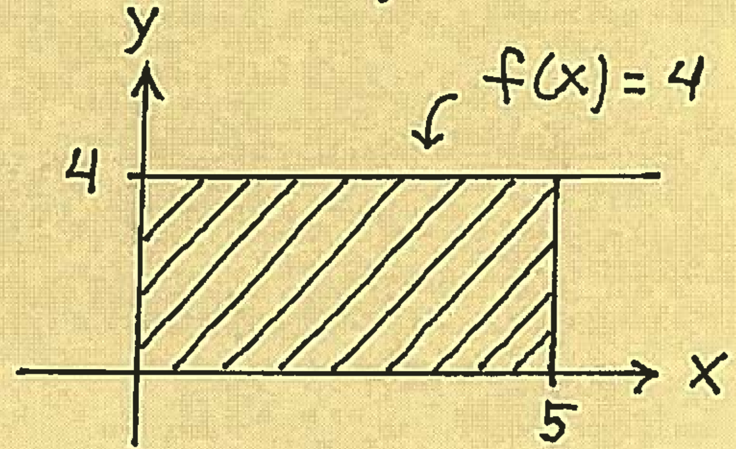
$$= \ln\left(\frac{3}{2}\right)$$

<u>QUESTION</u>: What is the meaning of a Definite Integral?

<u>ANSWER</u>: SEE next 4 pages.

# Areas of Geometric Shapes and Definite Integrals

<u>Example</u>:

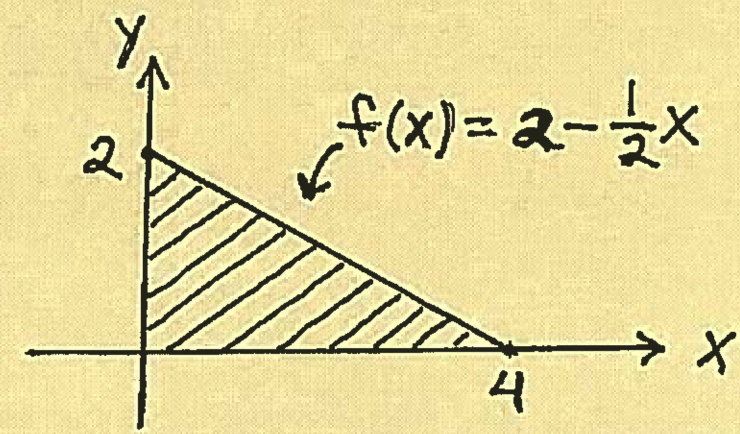



Area of ▭ $= (5)(4) = 20$ and

$$\int_0^5 f(x)\,dx = \int_0^5 4\,dx = 4x\Big|_0^5$$

$$= 4(5) - 4(0) = 20$$

<u>Example</u>:



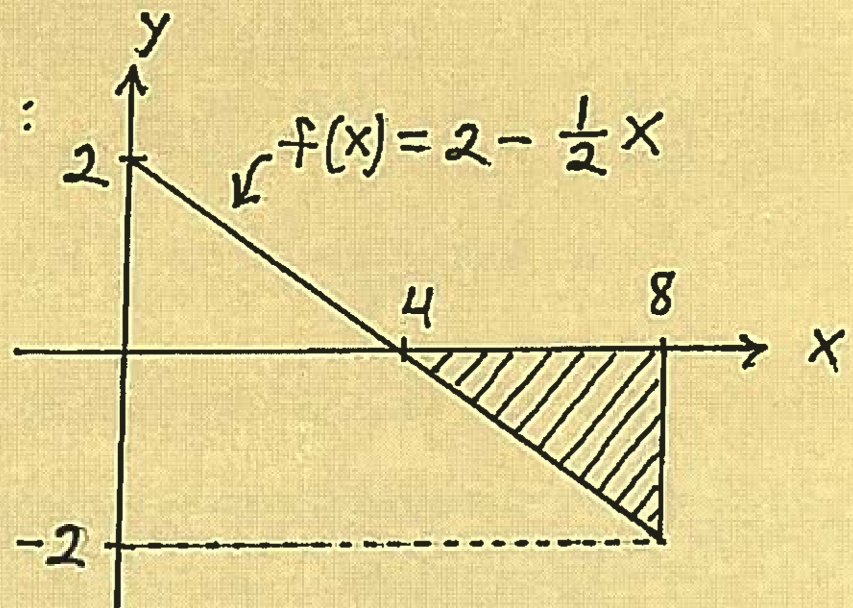

Area of $\triangle$ $= \frac{1}{2}(4)(2) = 4$ and

$$\int_0^4 f(x)\,dx = \int_0^4 \left(2 - \tfrac{1}{2}x\right) dx$$

$$= \left(2x - \tfrac{1}{4}x^2\right)\Big|_0^4 = \left(2(4) - \tfrac{1}{4}(4)^2\right) - \left(2(0) - \tfrac{1}{4}(0)^2\right)$$

$$= 8 - 4 - 0 = 4 \quad .$$

<u>Example</u>:



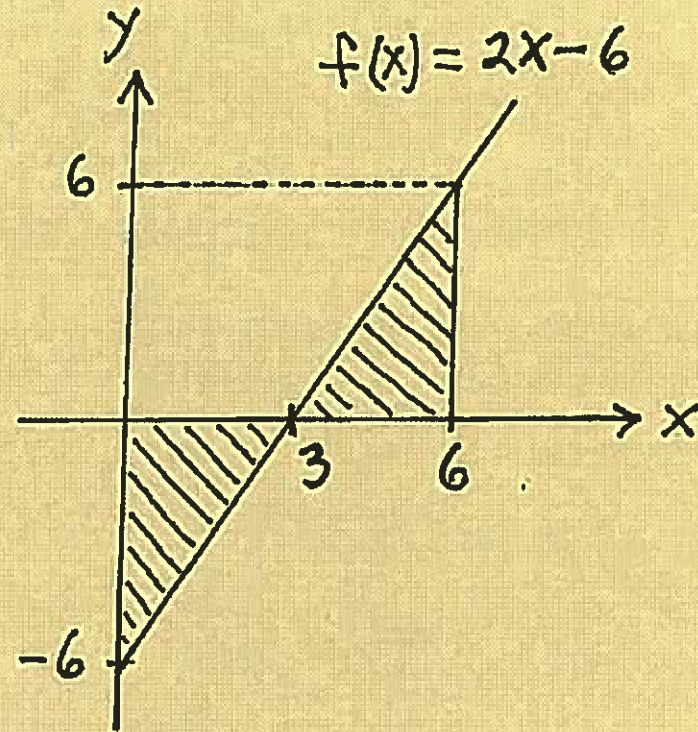

area of $\triangle$ $= \frac{1}{2}(4)(2) = 4$ , and

$$\int_4^8 \left(2 - \tfrac{1}{2}x\right) dx = \left(2x - \tfrac{1}{4}x^2\right)\Big|_4^8$$

$$= \left(2(8) - \tfrac{1}{4}(8)^2\right) - \left(2(4) - \tfrac{1}{4}(4)^2\right)$$

$$= 16 - 16 - (8 - 4) = -4 \; !$$

## Example:




Area of ◺ + ◿ $= \frac{1}{2}(3)(6) + \frac{1}{2}(3)(6) = 18$

and $\int_0^6 f(x)\,dx = \int_0^6 (2x-6)\,dx$

$= (x^2 - 6x)\Big|_0^6 = \left((6)^2 - 6(6)\right) - \left((0)^2 - 6(0)\right)$

$= 36 - 36 - 0 = 0$ !

## CONCLUSION :

I.) If $f(x) \geq 0$, then

$$\int_a^b f(x)\,dx = \text{AREA}$$

II.) If $f(x) \leq 0$, then

$$\int_a^b f(x)\,dx = -\,\text{AREA}$$