Properties of Definite Integrals, Even/Odd Functions

For Properties of and applications of Definite Integrals, SEE the next page.

Math 16B
Kouba
Properties of the Definite Integral

I.) Properties of the Definite Integral

a.) $\int_a^a f(x)\,dx = 0$

b.) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c.) $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

d.) $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

e.) If $f(x) \geq 0$ then $\int_a^b f(x)\,dx \geq 0$ (if $a < b$)

f.) If $f(x) \geq g(x)$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$ (if $a < b$)

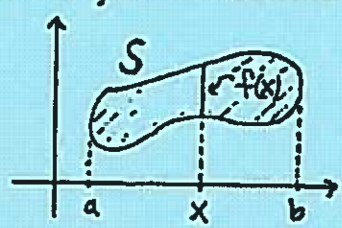
g.) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

h.) If $m \leq f(x) \leq M$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

II.) Applications of the Definite Integral

a.) Area of region : If $f(x)$ is the height of region S at x, then total area of S from a to b is

$$\text{AREA} = \int_a^b f(x)\,dx$$



b.) Mass of string : If $f(x)$ is the density (mass/length units) of string at x, then total mass of string from a to b is

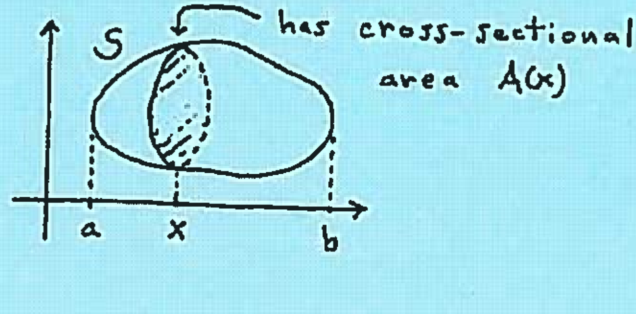
$$\text{MASS} = \int_a^b f(x)\,dx$$

c.) Distance traveled : If $f(t)$ is the speed of an object at time t, then total distance traveled from time a to time b is

$$\text{DISTANCE} = \int_a^b f(t)\,dt$$

d.) Volume of solid : If $A(x)$ is the cross-sectional area of a solid S at x, then total volume of S from a to b is

$$\text{VOLUME} = \int_a^b A(x)\,dx$$



Math 16B
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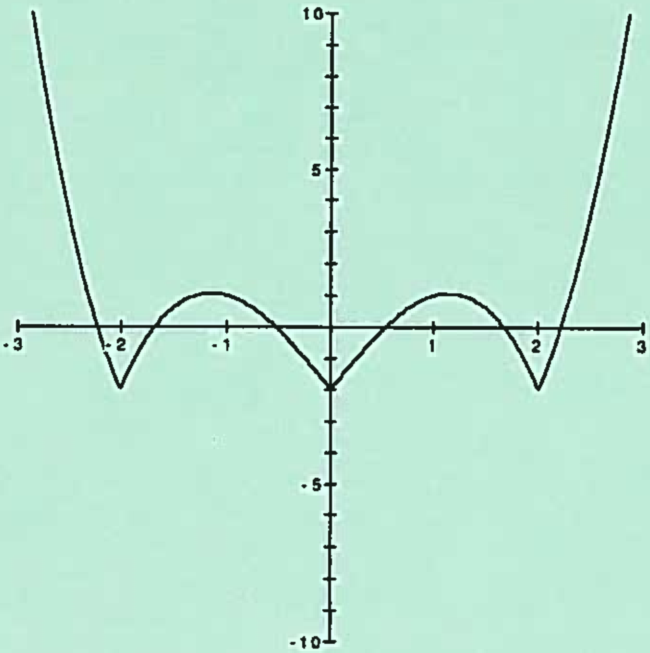
Even and Odd Functions

Knowing if a function is even or odd can sometimes lead to a relatively easy solution to a definite integral.

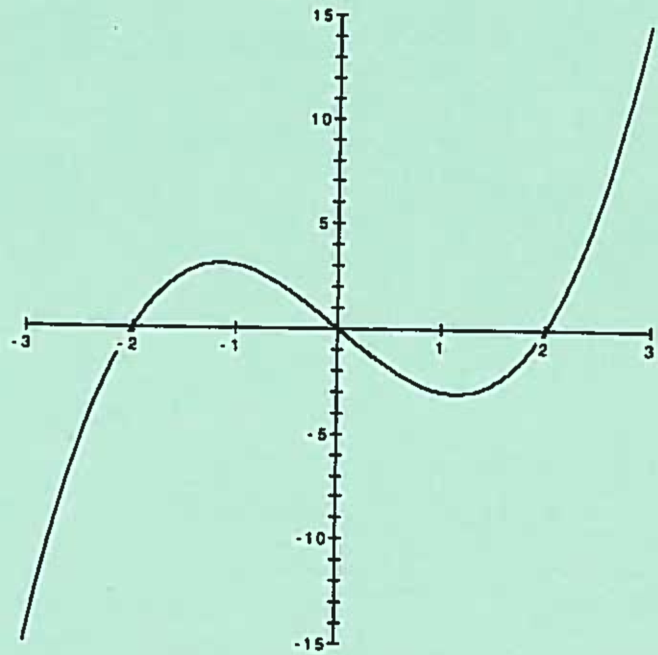
<u>DEFINITIONS</u> : Function f is *even* if $f(x) = f(-x)$. Function f is *odd* if $f(x) = -f(-x)$.

<u>EXAMPLE</u>:

f is even



f is odd



<u>REMARKS:</u>

I. If f is even then $\int_0^a f(x)\,dx = \int_{-a}^0 f(x)\,dx$ so that $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.

II. If f is odd then $\int_0^a f(x)\,dx = -\int_{-a}^0 f(x)\,dx$ so that $\int_{-a}^a f(x)\,dx = 0$.

<u>PROBLEM</u>: Show that $f(x) = x\sqrt{x^2 + \cos x}$ is an odd function , then evaluate the definite integral $\int_{-5}^{5} x\sqrt{x^2 + \cos x}\,dx$.

Even/Odd Problems

Example: Show that each of the following functions is EVEN, i.e., show that $f(-x) = f(x)$.

1.) $f(x) = 3x^4 - 2x^2 + 1$; then

$$f(-x) = 3(-x)^4 - 2(-x)^2 + 1$$
$$= 3x^4 - 2x^2 + 1$$
$$= f(x) \quad \text{, so } f \text{ is EVEN.}$$

2.) $f(x) = x^2 + \cos x$; then

$$f(-x) = (-x)^2 + \cos(-x)$$
$$= x^2 + \cos x$$
$$= f(x) \text{, so } f \text{ is EVEN.}$$

Example: Show that each of the following functions is ODD, i.e., show that $f(-x) = -f(x)$.

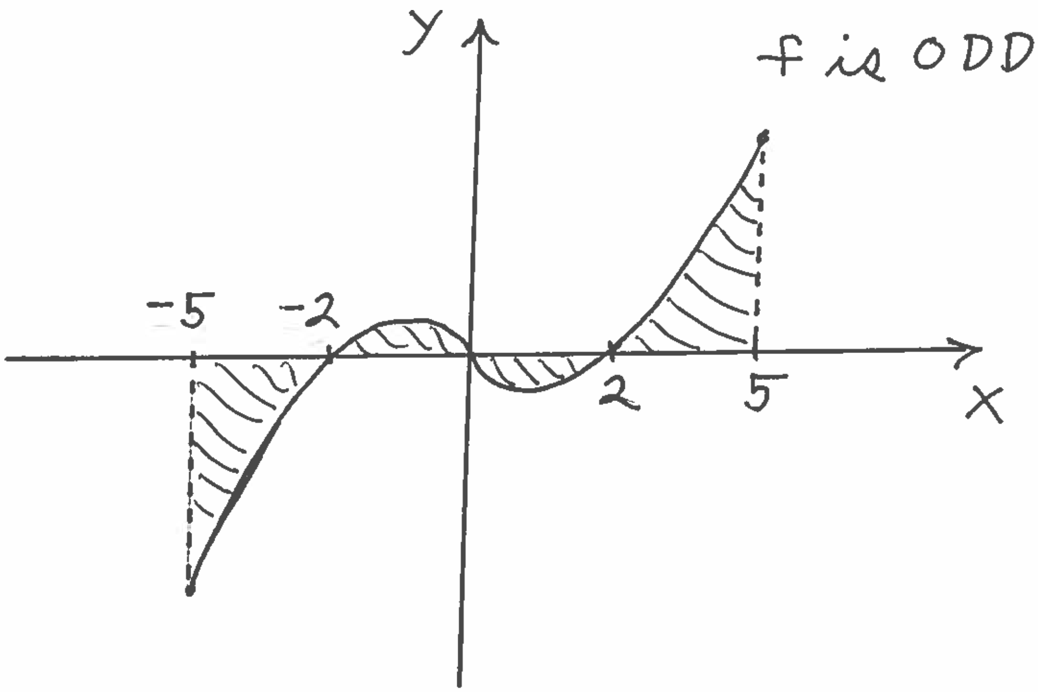
1.) $f(x) = x - 4x^3$; then

$$f(-x) = (-x) - 4(-x)^3$$
$$= -x - 4(-x^3)$$
$$= -x + 4x^3$$
$$= -(x - 4x^3)$$
$$= -f(x), \text{ so } f \text{ is ODD.}$$

2.) $f(x) = x^2 \sin x + x^3$; then

$$f(-x) = (-x)^2 \sin(-x) + (-x)^3$$
$$= x^2(-\sin x) - x^3$$
$$= -x^2 \sin x - x^3$$
$$= -(x^2 \sin x + x^3)$$
$$= -f(x), \text{ so } f \text{ is ODD.}$$

Example: assum f is ODD and $\int_{-2}^{5} f(x)\,dx = 4$. What is $\int_{-2}^{-5} 3f(x)\,dx$?



Given: $\int_{-2}^{5} f(x)\,dx = 4 \rightarrow$

$\underbrace{\int_{-2}^{2} f(x)\,dx}_{0} + \int_{2}^{5} f(x)\,dx = 4 \rightarrow$

f is ODD

$\int_{2}^{5} f(x)\,dx = 4 \rightarrow$

$\int_{-5}^{-2} f(x)\,dx = -4 \rightarrow$

$\int$ $\qquad \int_{-2}^{-5} f(x)\,dx = 4 \rightarrow$

$\int_{-2}^{-5} 3f(x)\,dx = 3\int_{-2}^{-5} f(x)\,dx = 3(4) = 12.$

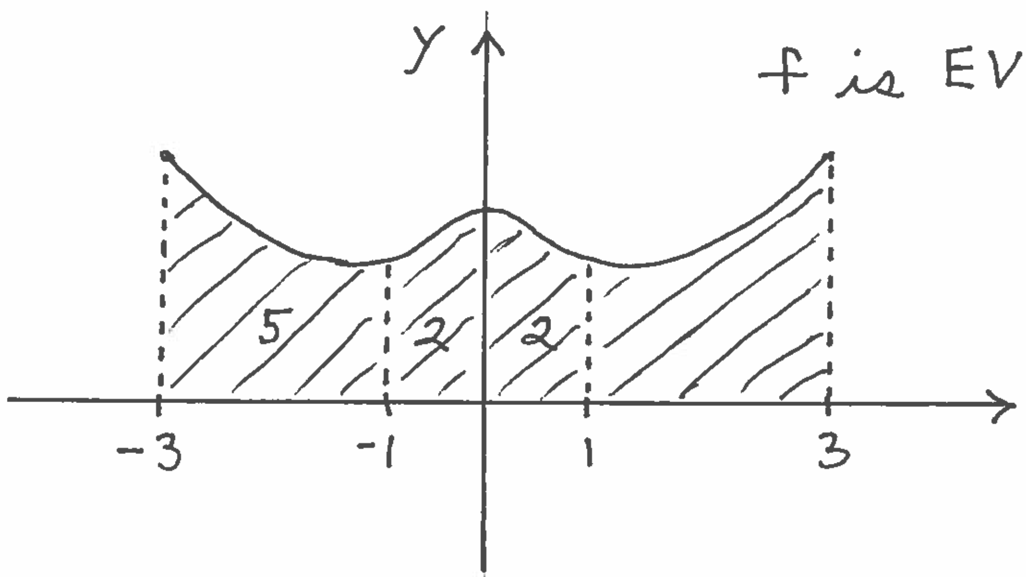
Example : assume that f is EVEN,

$\int_0^1 f(x)\,dx = 2$, and $\int_{-3}^{1} f(x)\,dx = 9$.

What is

a.) $\int_{-3}^{-1} f(x)\,dx$?

b.) $\int_{-3}^{3} f(x)\,dx$?



Given : $\int_0^1 f(x)\,dx = 2$ and

$\int_{-3}^{1} f(x)\,dx = 9 \longrightarrow$

$\int_{-1}^{0} f(x)\,dx = 2$; and

↖ f is EVEN

$$\int_{-3}^{1} f(x)\,dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = 9$$

$$\longrightarrow \quad \int_{-3}^{-1} f(x)\,dx + 2 + 2 = 9$$

a.) $\longrightarrow \quad \displaystyle\int_{-3}^{-1} f(x)\,dx = 5$; *then*

$\displaystyle\int_{1}^{3} f(x)\,dx = 5$ *since* f *is* EVEN ; *and*

$$\int_{-3}^{3} f(x)\,dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$$

$$= 5 + 2 + 2 + 5 = 14 \text{; i.e.,}$$

b.) $\displaystyle\int_{-3}^{3} f(x)\,dx = 14$.