Math 16B
Section 6.5

Simpson's Rule – Estimating the Value of Definite Integrals (again)

RECALL: 1.) Midpoint Rule
2.) Trapezoidal Rule

(SEE NEXT PAGE.)

I.) Midpoint Rule uses rectangles to estimate area.

II.) Trapezoidal Rule uses trapezoids to estimate area.

III.) Simpson's Rule uses regions topped by parabolas to estimate area.

Math 16B
Kouba
Estimating the Value of a Definite Integral

Suppose that the integral $\int_a^b f(x)\,dx$ is too difficult (or impossible) to compute, or that you are simply required to estimate its exact value. The following three methods offer three different ways to compute an estimate.

1.) MIDPOINT RULE

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \frac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval and let the sampling points $c_1, c_2, c_3, ..., c_n$ be the MIDPOINTS of these subintervals.

c.) The Midpoint Estimate for $\int_a^b f(x)\,dx$ is

$$M_n = h\,[f(c_1) + f(c_2) + f(c_3) + ... + f(c_n)] \ .$$

d.) The Absolute Error is $|E_n| \le (b-a)h\left\{ \max_{a \le x \le b} |f'(x)| \right\}$.

2.) TRAPEZOIDAL RULE

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \frac{b-a}{n}$.

b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Trapezoidal Estimate for $\int_a^b f(x)\,dx$ is

$$T_n = \frac{h}{2}\,[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)] \ .$$

d.) The Absolute Error is $|E_n| \le (b-a)\frac{h^2}{12}\left\{ \max_{a \le x \le b} |f''(x)| \right\}$.

3.) SIMPSON'S RULE (NOTE: For this method n MUST be an even integer !)

a.) Divide the interval $[a, b]$ into n equal parts, each of length $h = \frac{b-a}{n}$.

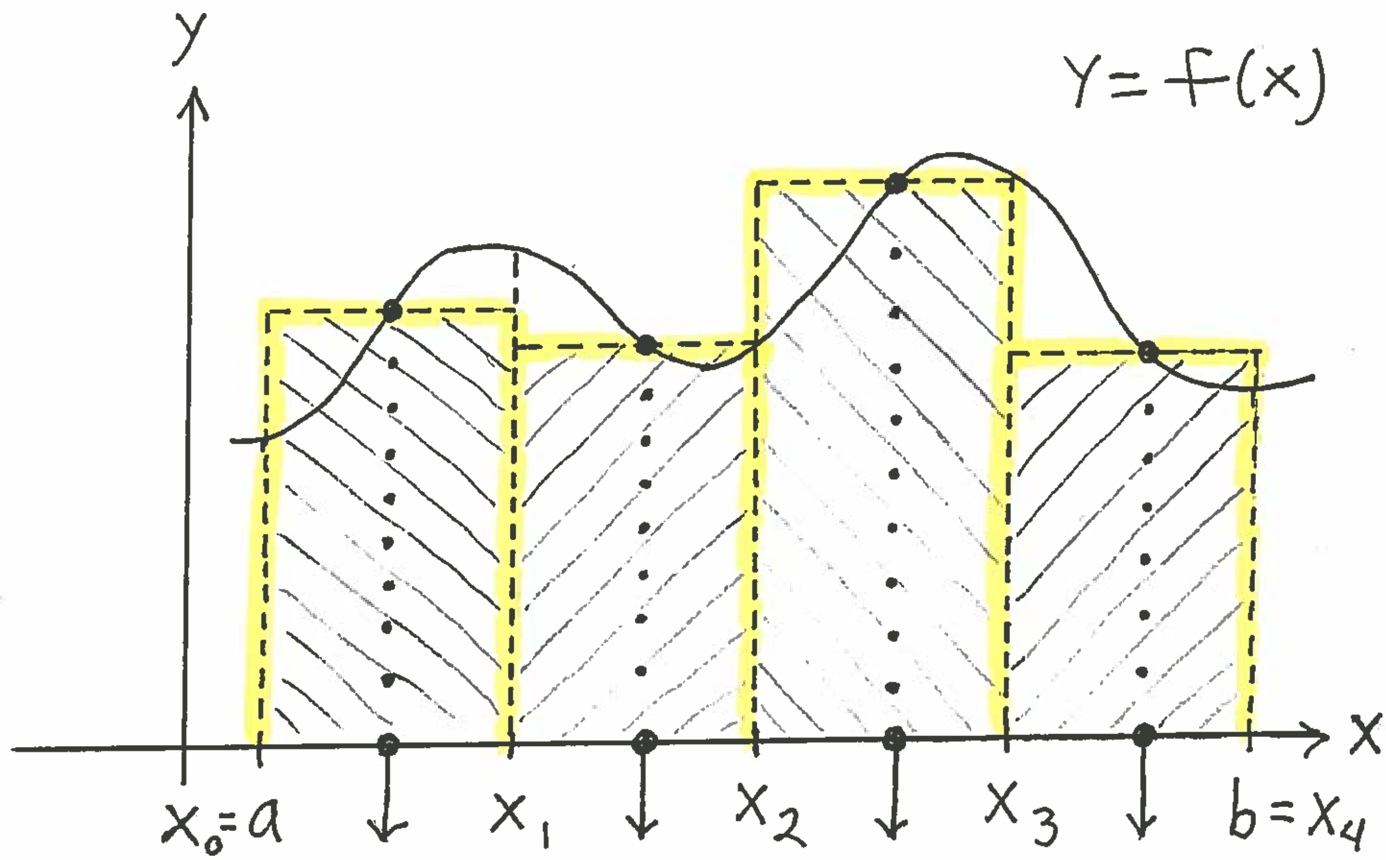
b.) Let $a = x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n = b$ be the partition of the interval.

c.) The Simpson Estimate for $\int_a^b f(x)\,dx$ is

$$S_n = \frac{h}{3}\,[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+4f(x_{n-3})+2f(x_{n-2})+4f(x_{n-1})+f(x_n)] \ .$$

d.) The Absolute Error is $|E_n| \le (b-a)\frac{h^4}{180}\left\{ \max_{a \le x \le b} |f^{(4)}(x)| \right\}$.

Midpoint Rule with $n=4$:

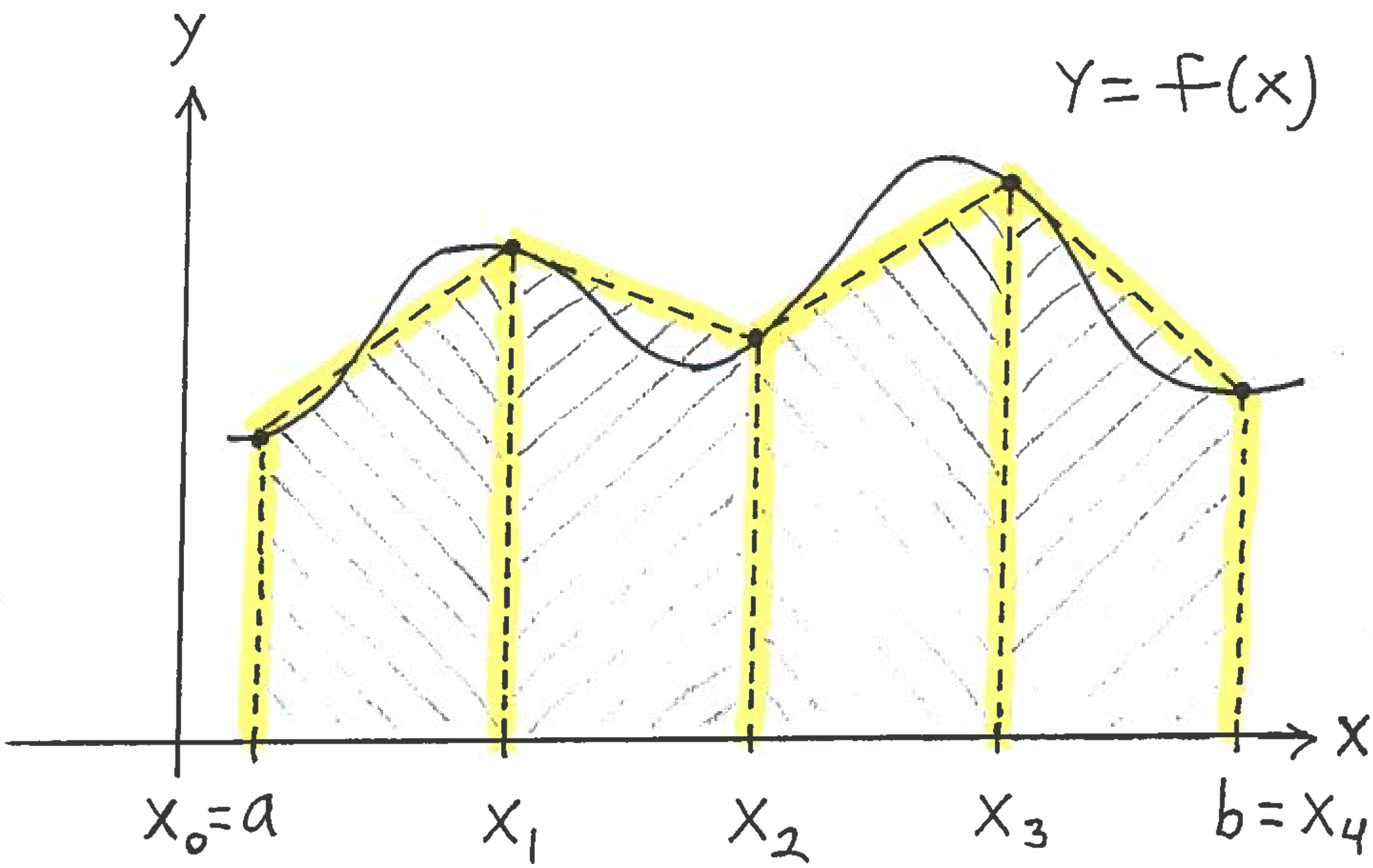


midpoints: c_1 c_2 c_3 c_4

$$\int_a^b f(x)\,dx \approx M_4$$

$$= h\left[f(c_1) + f(c_2) + f(c_3) + f(c_4)\right]$$

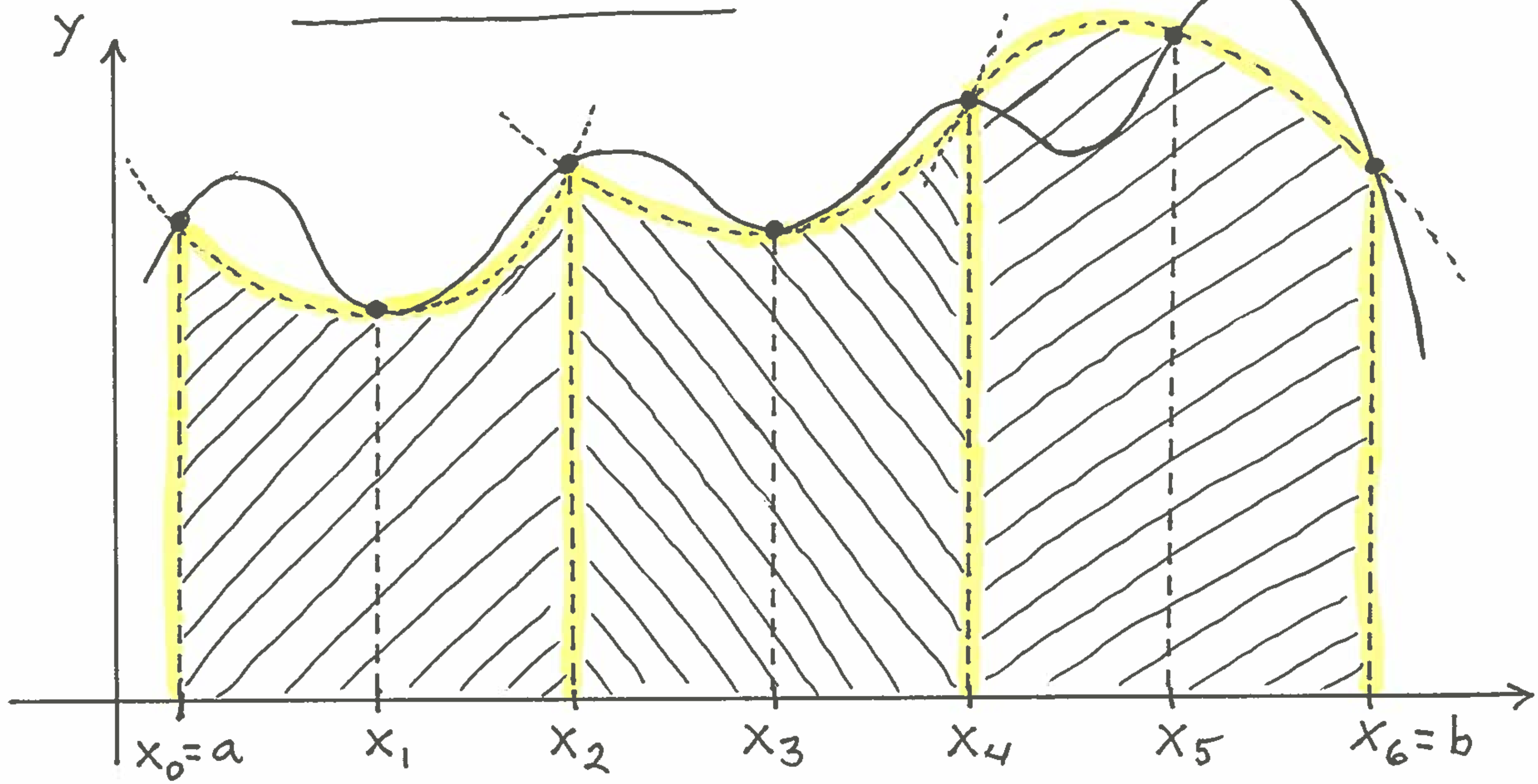
Trapezoidal Rule with $n = 4$:



$$\int_a^b f(x)\,dx \approx T_4$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

Simpson's Rule with $n = 6$



$$\int_a^b f(x) \approx S_6$$

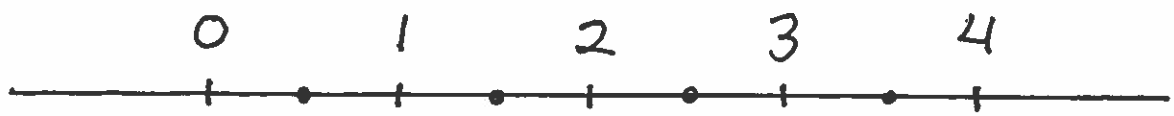
$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6) \right]$$

<u>Example</u>: Estimate the value of $\int_0^4 \sqrt{2+\sqrt{x}}\,dx$ using

1.) Midpoint Rule with $n=4$.

2.) Trapezoidal Rule with $n=4$.

3.) Simpson's Rule with $n=4$.



midpoints: $\frac{1}{2}$ $\frac{3}{2}$ $\frac{5}{2}$ $\frac{7}{2}$

$f(x)=\sqrt{2+\sqrt{x}}\,,\; n=4\,,\; h=\frac{4-0}{4}=1$

1.) $M_4 = h\left[f\left(\tfrac{1}{2}\right)+f\left(\tfrac{3}{2}\right)+f\left(\tfrac{5}{2}\right)+f\left(\tfrac{7}{2}\right)\right]$

$= 1\cdot\left[\sqrt{2+\sqrt{1/2}}+\sqrt{2+\sqrt{3/2}}+\sqrt{2+\sqrt{5/2}}+\sqrt{2+\sqrt{7/2}}\right]$

≈ 7.3009

2.) $T_4 = \frac{h}{2}\left[f(0)+2f(1)+2f(2)+2f(3)+f(4)\right]$

$= \frac{1}{2}\left[\sqrt{2}+2\sqrt{3}+2\sqrt{2+\sqrt{2}}+2\sqrt{2+\sqrt{3}}+2\right]$

≈ 7.2188

3.) $S_4 = \frac{h}{3}\left[f(0)+4f(1)+2f(2)+4f(3)+f(4)\right]$

$= \frac{1}{3}\left[\sqrt{2}+4\sqrt{3}+2\sqrt{2+\sqrt{2}}+4\sqrt{2+\sqrt{3}}+2\right]$

≈ 7.2551

<u>CALCULATOR CHECK</u>:

$\int_0^4 \sqrt{2+\sqrt{x}}\,dx \approx 7.2836$

Using Absolute Error Formulas

First, let's play the BIG/SMALL GAME.

FACT: To make a FRACTION BIGGER do one or both of the following.

I.) Make the TOP BIGGER.

II.) Make the BOTTOM SMALLER.

Example: Make each fraction as big as possible on the interval $-1 \le x \le 2$.

1.) $\dfrac{|x-3|}{7} \le \dfrac{|(-1)-3|}{7} = \dfrac{4}{7}$

2.) $\dfrac{3}{|x-4|} \le \dfrac{3}{|(2)-4|} = \dfrac{3}{2}$

3.) $\dfrac{|x^2+1|}{|x+3|} \le \dfrac{|2^2+1|}{|(-1)+3|} = \dfrac{5}{2}$

4.) $\dfrac{|3x-4|}{|4x-9|} \le \dfrac{|3(-1)-4|}{|4(2)-9|} = 7$

5.) $\dfrac{|x+3|}{|x-3|} \le \dfrac{|(2)+3|}{|(2)-3|} = 5$

<u>Question</u>: Without knowing the value $\int_a^b f(x)\,dx$, how can we determine if a Midpoint, Trapezoidal, or Simpson Estimate is a "good" one?

<u>Answer</u>: Use an Absolute Error Formula.

<u>Example</u>: If T_{20}, the Trapezoidal Rule with $n=20$, is used to estimate the value of $\int_0^1 \frac{1}{x^2+1}\,dx$, then the absolute

Error is defined to be

$$|E_{20}| = \left| \int_0^1 \frac{1}{x^2+1}\,dx - T_{20} \right| .$$

Let's estimate the value of $|E_{20}|$ using formula 2.) d.) :

$$f(x) = \frac{1}{x^2+1} = (x^2+1)^{-1} \xrightarrow{D}$$

$$f'(x) = -(x^2+1)^{-2} \cdot 2x = \frac{-2x}{(x^2+1)^2} \xrightarrow{D}$$

$$f''(x) = \frac{(x^2+1)^2(-2) - (-2x)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4}$$

$$= \frac{-2(x^2+1)\left[(x^2+1) - 4x^2\right]}{(x^2+1)^4}$$

$$= \frac{-2\left[1-3x^2\right]}{(x^2+1)^3} \text{ ; then}$$

$$\max_{0 \le x \le 1} |f''(x)| = \max_{0 \le x \le 1} \left| \frac{-2\,[1-3x^2]}{(x^2+1)^3} \right|$$

$$= \max_{0 \le x \le 1} \frac{2\,|1-3x^2|}{(x^2+1)^3}$$

BIG/SMALL GAME

$$\le \frac{2\,|1-3(1)^2|}{((0)^2+1)^3}$$

$$= 4 \quad ; \quad \text{now}$$

$$|E_n| \le (b-a)\frac{h^2}{12}\left\{\max_{a \le x \le b} |f''(x)|\right\}$$

$$= (1-0)\frac{\left(\frac{1-0}{20}\right)^2}{12}\left\{\max_{0 \le x \le 1} |f''(x)|\right\}$$

$$= \frac{1}{(20)^2\,12}\{4\}$$

$$= \frac{1}{1200}$$

$$\approx 0.000833$$

<u>Example</u>: What should n be so that the Simpson Estimate, S_n, estimates the exact value of $\int_0^3 (2x+4)^{5/2}\,dx$ with Absolute Error of at most 0.00001 ?

$$f(x) = (2x+4)^{5/2} \xrightarrow{D}$$

$$f'(x) = \frac{5}{2}\cdot 2(2x+4)^{3/2} = 5(2x+4)^{3/2} \xrightarrow{D}$$

$$f''(x) = 5\cdot\frac{3}{2}\cdot 2(2x+4)^{1/2} = 15(2x+4)^{1/2} \xrightarrow{D}$$

$$f'''(x) = 15\cdot\frac{1}{2}\cdot 2(2x+4)^{-1/2} = 15(2x+4)^{-1/2} \xrightarrow{D}$$

$$f^{(4)}(x) = 15\cdot\frac{-1}{2}\cdot 2(2x+4)^{-3/2} = -15(2x+4)^{-3/2}, \text{ i.e.,}$$

$$f^{(4)}(x) = \frac{-15}{(2x+4)^{3/2}}, \text{ so that}$$

$$\max_{0\le x\le 3}\left|\frac{-15}{(2x+4)^{3/2}}\right| \le \frac{15}{(2(0)+4)^{3/2}} = \frac{15}{8};$$

the Absolute Error for Simpson's Rule is

$$|E_n| \le (b-a)\frac{h^4}{180}\left\{\max_{a\le x\le b}\left|f^{(4)}(x)\right|\right\}$$

$$= (3-0)\,\frac{\left(\frac{3-0}{n}\right)^4}{180}\left\{\max_{0\le x\le 3}\left|f^{(4)}(x)\right|\right\}$$

$$= \frac{1}{60}\cdot\frac{81}{n^4}\left\{\frac{15}{8}\right\}$$

$$= \frac{81}{32}\cdot\frac{1}{n^4}\ ,\ \text{i.e.,}$$

$$|E_n| \le \frac{81}{32}\cdot\frac{1}{n^4} \le 0.00001 \longrightarrow$$

$$n^4 \ge \frac{81}{32(0.00001)} \longrightarrow$$

$$n^4 \ge 253{,}125 \longrightarrow$$

$$n \ge (253{,}125)^{1/4} \approx 22.4\ ,\ \text{so}$$

choose $\boxed{n = 24}$.