Math 16B
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO TAKE AN EXAM FOR ANOTHER PERSON. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 6 pages, including the cover page.

6. Include units on answers where units are appropriate.

7. You will be graded on proper use of integral and derivative notation.

8. You have until 2:00 p.m. sharp to finish the exam.

9. You may use the following trig identities :

a.) $\sin^2\theta + \cos^2\theta = 1$
b.) $1 + \tan^2\theta = \sec^2\theta$
c.) $\sin 2\theta = 2\sin\theta\cos\theta$
d.) $\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

10. Absolute Error for Trapezoidal Rule is : $|E_n| \le (b-a)\dfrac{h^2}{12}\left\{\max_{a\le x\le b}|f''(x)|\right\}$,

Absolute Error for Simpson's Rule is : $|E_n| \le (b-a)\dfrac{h^4}{180}\left\{\max_{a\le x\le b}|f^{(4)}(x)|\right\}$.

1.) (9 pts. each) Use partial fractions for each indefinite integral.

a.) $\int \frac{x+3}{x^2-x-2}\,dx = \int \frac{x+3}{(x-2)(x+1)}\,dx = \int \left[\frac{A}{x-2} + \frac{B}{x+1}\right] dx$

$\Big(x+3 = A(x+1) + B(x-2)$

$\underline{\text{Let } x=2}: \quad 5 = 3A \rightarrow \quad A = 5/3$

$\underline{\text{Let } x=-1}: \quad 2 = -3B \rightarrow \quad B = -2/3 \Big)$

$= \int \left[\frac{5/3}{x-2} + \frac{-2/3}{x+1}\right] dx$

$= \frac{5}{3}\ln|x-2| - \frac{2}{3}\ln|x+1| + C$

b.) $\int \frac{x^2+x-1}{x^3-x^2}\,dx = \int \frac{x^2+x-1}{x^2(x-1)}\,dx = \int \left[\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}\right] dx$

$\Big(x^2+x-1 = Ax(x-1) + B(x-1) + Cx^2$

$\underline{\text{Let } x=0}: \quad -1 = -B \rightarrow \quad B = 1$

$\underline{\text{Let } x=1}: \quad 1 = C$

$\underline{\text{Let } x=2}: \quad 5 = 2A + (1)(1) + (1)(4) \rightarrow 2A = 0 \rightarrow A = 0 \Big)$

$= \int \left[\frac{1}{x^2} + \frac{1}{x-1}\right] dx = \frac{-1}{x} + \ln|x-1| + C$

2.) (9 pts. each) Determine the value of each improper integral.

a.) $\int_0^\infty e^{-2x}\, dx = \lim_{A\to\infty} \int_0^A e^{-2x}\, dx$

$= \lim_{A\to\infty} \frac{-1}{2} e^{-2x}\Big|_0^A = \lim_{A\to\infty} \left(\frac{-1}{2} e^{-2A} - \frac{-1}{2} e^0\right)$

$= \lim_{A\to\infty} \left(\underbrace{\frac{-1}{2e^{2A}}}_{0} + \frac{1}{2}\right) = \frac{1}{2}$

b.) $\int_1^\infty \frac{1}{(x-1)^{2/3}}\, dx = \int_1^2 (x-1)^{-2/3}\, dx + \int_2^\infty (x-1)^{-2/3}\, dx = B + C$;

$B = \lim_{A\to 1^+} \int_A^2 (x-1)^{-2/3}\, dx = \lim_{A\to 1^+} 3(x-1)^{1/3}\Big|_A^2$

$= \lim_{A\to 1^+} \left(3(1)^{1/3} - 3(A-1)^{1/3}\right) = 3 - 3(0) = 3$;

$C = \lim_{A\to\infty} \int_2^A (x-1)^{-2/3}\, dx = \lim_{A\to\infty} 3(x-1)^{1/3}\Big|_2^A$

$= \lim_{A\to\infty} \left(3(A-1)^{1/3} - 3(1)^{1/3}\right) = \infty - 3 = \infty$, so

$\int_1^\infty (x-1)^{-2/3}\, dx = B + C = 3 + \infty = \infty$.

c.) $\int_4^\infty \frac{1}{x^2-9}\, dx = \int_4^\infty \frac{1}{(x-3)(x+3)}\, dx = \int_4^\infty \left[\frac{A}{x-3} + \frac{B}{x+3}\right] dx$

($1 = A(x+3) + B(x-3)$; let $x = 3$: $1 = 6A \to A = 1/6$
let $x = -3$: $1 = -6B \to B = -1/6$)

$= \lim_{A\to\infty} \int_4^A \left[\frac{1/6}{x-3} + \frac{-1/6}{x+3}\right] dx = \lim_{A\to\infty} \left[\frac{1}{6}\ln|x-3| - \frac{1}{6}\ln(x+3)\right]\Big|_4^A$

$= \lim_{A\to\infty} \left[\left(\frac{1}{6}\ln|A-3| - \frac{1}{6}\ln|A+3|\right) - \left(\frac{1}{6}\underbrace{\ln 1}_{0} - \frac{1}{6}\ln 7\right)\right]$

$= \lim_{A\to\infty} \frac{1}{6}\ln\left|\frac{A-3}{A+3}\right| + \frac{1}{6}\ln 7$

$= \lim_{A\to\infty} \frac{1}{6}\ln\left|\frac{A-3}{A+3}\cdot\frac{1/A}{1/A}\right| + \frac{1}{6}\ln 7 = \lim_{A\to\infty} \frac{1}{6}\ln\left|\frac{1-3/A}{1+3/A}\right| + \frac{1}{6}\ln 7$

$= \frac{1}{6}\ln\left|\frac{1-0}{1+0}\right| + \frac{1}{6}\ln 7 = \frac{1}{6}\cancel{\ln 1} + \frac{1}{6}\ln 7 = \frac{1}{6}\ln 7$

3.) (9 pts.) Compute S_6, the Simpson Estimate using $n = 6$, for $\int_{-3}^{3} \sqrt{\frac{x+3}{x+4}}\, dx$.

$-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3$

$f(x) = \sqrt{\frac{x+3}{x+4}}$, $h = \frac{3-(-3)}{6} = 1$

$S_6 = \frac{h}{3}\left[f(-3) + 4f(-2) + 2f(-1) + 4f(0) + 2f(1) + 4f(2) + f(3)\right]$

$= \frac{1}{3}\left[0 + 4\sqrt{\frac{1}{2}} + 2\sqrt{\frac{2}{3}} + 4\sqrt{\frac{3}{4}} + 2\sqrt{\frac{4}{5}} + 4\sqrt{\frac{5}{6}} + \sqrt{\frac{6}{7}}\right]$

≈ 4.764

4.) (9 pts.) Use $\int \sqrt{u^2 \pm a^2}\, du = (1/2)\left(u\sqrt{u^2 \pm a^2} \pm a^2 \ln|u + \sqrt{u^2 \pm a^2}|\right) + C$ to integrate $\int \sqrt{x^2 - 2x + 5}\, dx$.

$\int \sqrt{x^2 - 2x + 5}\, dx = \int \sqrt{(x^2 - 2x + 1) + 4}\, dx = \int \sqrt{(x-1)^2 + 2^2}\, dx$

(Let $u = x - 1 \xrightarrow{D} du = 1\,dx$, $a = 2$)

$= \int \sqrt{u^2 + 2^2}\, du = \frac{1}{2}\left(u\sqrt{u^2 + 2^2} + 2^2 \ln|u + \sqrt{u^2 + 2^2}|\right) + c$

$= \frac{1}{2}\left((x-1)\sqrt{(x-1)^2 + 2^2} + 2^2 \ln|(x-1) + \sqrt{(x-1)^2 + 2^2}|\right) + c$

5.) (9 pts. each) Determine the following indefinite integrals.

a.) $\displaystyle\int \frac{\tan 2x}{\sec 2x}\,dx = \int \frac{\frac{\sin 2x}{\cos 2x}}{\frac{1}{\cos 2x}}\,dx = \int \frac{\sin 2x}{\cancel{\cos 2x}} \cdot \frac{\cancel{\cos 2x}}{1}\,dx$

$= \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$

(Let $u = (\ln x)^2$, $dv = dx$

b.) $\displaystyle\int (\ln x)^2\,dx$ $\rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$)

$= x(\ln x)^2 - 2\int \ln x\,dx$ (Let $u = \ln x$, $dv = dx$ $\rightarrow du = \frac{1}{x}\,dx$, $v = x$)

$= x(\ln x)^2 - 2\left[x\ln x - \int 1\,dx\right]$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

c.) $\displaystyle\int e^x \sin x\,dx$ (HINT : Use integration by parts twice with a twist.)

$= -e^x\cos x - -\int e^x \cos x\,dx$ (Let $u = e^x$, $dv = \sin x\,dx$ $\rightarrow du = e^x\,dx$, $v = -\cos x$)

(Let $u = e^x$, $dv = \cos x\,dx$ $\rightarrow du = e^x\,dx$, $v = \sin x$)

$= -e^x\cos x + \left[e^x \sin x - \int e^x \sin x\,dx\right]$

$(A = B + C - A) \rightarrow$

$2\int e^x \sin x\,dx = -e^x\cos x + e^x \sin x + C \rightarrow$

$\int e^x \sin x\,dx = -\frac{1}{2}e^x\cos x + \frac{1}{2}e^x \sin x + C$

7.) (10 pts.) What should n be in order that the Trapezoidal Rule estimate the exact value of the following definite integral with absolute error at most 0.0005 ? See the Error Formula on the front of this exam. Assume that the second derivative of $f(x) = \ln\left(1+\frac{1}{x}\right)$ is $f''(x) = \dfrac{2x+1}{x^2(x+1)^2}$. $\displaystyle\int_1^2 \ln\left(1+\frac{1}{x}\right)\,dx$

$$\max_{1\le x\le 2} |f''(x)| = \max_{1\le x\le 2} \frac{|2x+1|}{x^2(x+1)^2} \le \frac{2(2)+1}{(1)^2(1+1)^2} = \frac{5}{4},$$

$h = \frac{2-1}{n} = \frac{1}{n}$; then

$$|E_n| \le (2-1)\cdot\frac{\left(\frac{1}{n}\right)^2}{12}\cdot\left\{\frac{5}{4}\right\} = \frac{5}{48}\cdot\frac{1}{n^2} \le 0.0005 \rightarrow$$

$$n^2 \ge \frac{5}{48(0.0005)} \rightarrow$$

$$n \ge \sqrt{\frac{5}{48(0.0005)}} \approx 14.4 \text{ so choose } n = 15$$

EXTRA CREDIT PROBLEM– The following problem is worth 10 points. This problem is OPTIONAL.

1.) $\displaystyle\int \frac{1}{e^{2x}+e^x}\,dx = \int \frac{1}{e^x(e^x+1)}\,dx$ (Let $u = e^x + 1 \xrightarrow{D}$ $du = e^x\,dx$ and $e^x = u-1 \rightarrow du = (u-1)\,dx \rightarrow \frac{1}{u-1}\,du = dx$)

$$= \int \frac{1}{(u-1)u}\cdot\frac{1}{u-1}\,du = \int \frac{1}{u(u-1)^2}\,du = \int\left[\frac{A}{u}+\frac{B}{u-1}+\frac{C}{(u-1)^2}\right]du$$

($1 = A(u-1)^2 + Bu(u-1) + Cu$; Let $u=1$: $1 = C$
Let $u=0$: $1 = A$
Let $u=2$: $1 = (1)(1) + 2B + (1)(2)$
$\rightarrow 2B = -2 \rightarrow B = -1$)

$$= \int\left[\frac{1}{u}+\frac{-1}{u-1}+\frac{1}{(u-1)^2}\right]du$$

$$= \ln|u| - \ln|u-1| - \frac{1}{u-1} + C$$

$$= \ln|e^x+1| - \ln e^x - \frac{1}{(e^x+1)-1} + C = \ln|e^x+1| - x - \frac{1}{e^x} + C$$