Math 16C
Kouba
Lagrange Multiplier
Problem with Two Constraints

Ex: Minimize and maximize the function
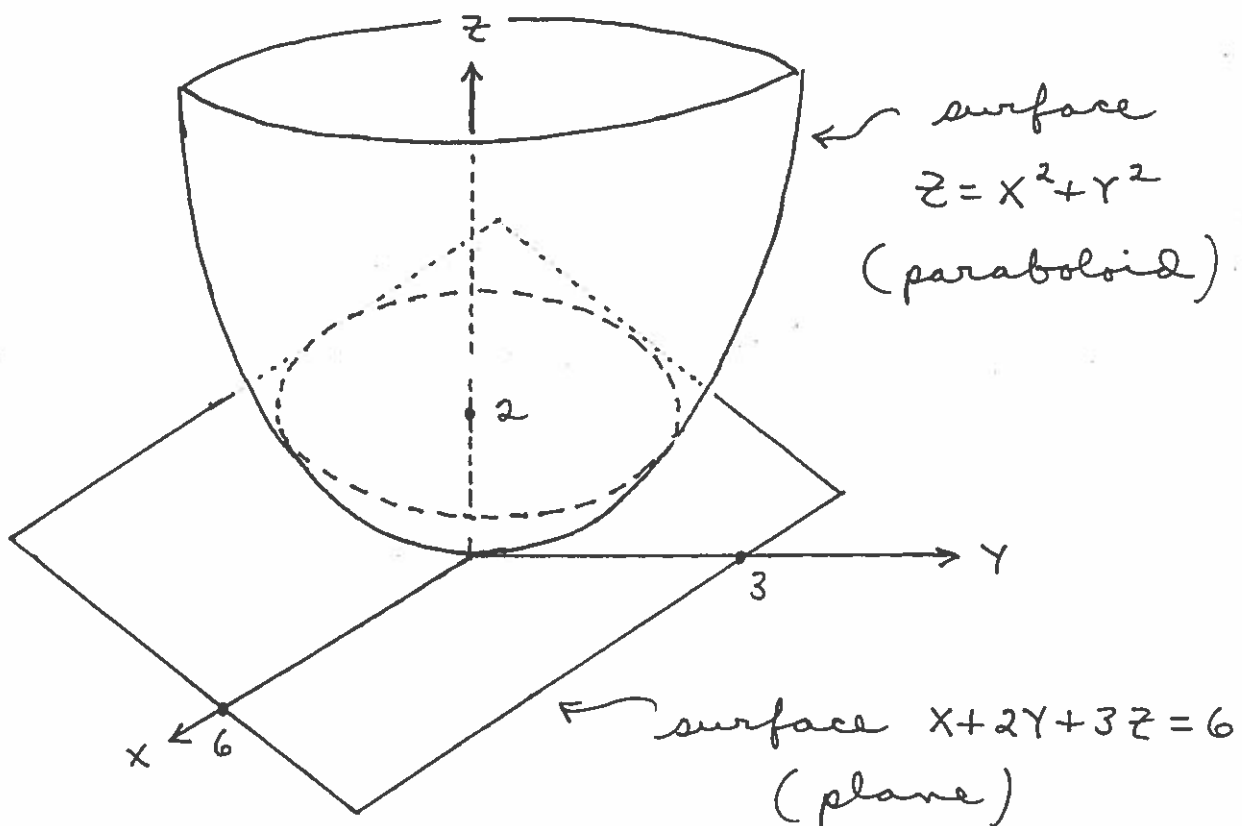
$T = 50 + X + 2Y + Z^2$ subject to the two

constraints $Z = X^2 + Y^2$

and $X + 2Y + 3Z = 6$:

The two constraints taken together represent the intersection of a plane and a paraboloid. This intersection is represented by the dotted "ring" in the diagram below.

surface $Z = X^2 + Y^2$ (paraboloid)

surface $X + 2Y + 3Z = 6$ (plane)

If we assume that the T equation
measures the temperature at point
(x, y, z), then for this problem we are
seeking the points on the "ring" where
the temperature is hottest and coldest.
The solution using Lagrange multipliers
follows :

Let $F = (50 + x + 2y + z^2) - \lambda(z - x^2 - y^2) - \mu(6 - x - 2y - 3z)$
then

$$F_x = 1 + 2\lambda x + \mu = 0$$

$$F_y = 2 + 2\lambda y + 2\mu = 0$$

$$F_z = 2z - \lambda + 3\mu = 0$$

$$F_\lambda = x^2 + y^2 - z = 0$$

$$F_\mu = x + 2y + 3z - 6 = 0 \quad ;$$

solve the first three equations for μ
getting

$$\left. \begin{array}{l} \mu = -1 - 2\lambda x \\ \mu = -1 - \lambda y \\ \mu = -\frac{2}{3}z + \frac{1}{3}\lambda \end{array} \right\} \quad 2\lambda x = \lambda y \rightarrow 2\lambda x - \lambda y = 0 \rightarrow$$

$$\lambda(2x - y) = 0 \rightarrow \boxed{\lambda = 0} \text{ or } \boxed{y = 2x} \quad ;$$

<u>case 1.</u> If $\lambda = 0$ then $\mu = -1$ and $-1 = -\frac{2}{3}z \rightarrow$
$z = \frac{3}{2}$. Substitute into the fourth and
fifth equations getting

$$X^2 + Y^2 - \frac{3}{2} = 0 \quad \longleftarrow$$
$$X + 2Y + \frac{9}{2} - 6 = 0 \quad \longrightarrow \quad X = \frac{3}{2} - 2Y$$

$$\left(\frac{3}{2} - 2Y\right)^2 + Y^2 - \frac{3}{2} = 0 \quad \longrightarrow$$
$$\frac{9}{4} - 6Y + 4Y^2 + Y^2 - \frac{3}{2} = 0 \quad \longrightarrow$$
$$5Y^2 - 6Y + \frac{3}{4} = 0 \quad \longrightarrow$$
$$20Y^2 - 24Y + 3 = 0 \quad \longrightarrow$$

$$Y = \frac{24 \pm \sqrt{24^2 - 240}}{40} = \frac{24 \pm \sqrt{336}}{40} = 1.06 \text{ or } 0.14$$

so critical points and temperatures are

$$X = -0.62, \; Y = 1.06, \; Z = 1.5 \quad \text{and } T = 53.75°$$
$$X = 1.22, \; Y = 0.14, \; Z = 1.5 \quad \text{and } T = 53.75°.$$

<u>case 2.</u> If $Y = 2X$ then fourth and fifth equations become

$$\left. \begin{array}{l} X^2 + (2X)^2 - Z = 0 \\ X + 2(2X) + 3Z - 6 = 0 \end{array} \right\} \quad \left. \begin{array}{l} Z = 5X^2 \\ 5X + 3Z = 6 \end{array} \right\} \quad 15X^2 + 5X - 6 = 0 \; \Rightarrow$$

$$X = \frac{-5 \pm \sqrt{25 + 360}}{30} = 0.49 \text{ or } -0.82$$

so critical points and temperatures are

$$X = -0.82, \; Y = -1.64, \; Z = 3.36 \quad \text{and } T = 57.19°$$
$$X = 0.49, \; Y = 0.98, \; Z = 1.20 \quad \text{and } T = 53.89°.$$

The maximum value of T is $57.19°$.
The minimum value of T is $53.75°$.