Math 17B
Kouba
Differentiating an Inverse Trig Function

Trig Function	Domain Restriction	Inverse Function	Derivative of Inverse
$y = \tan x$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan x$	$y' = \frac{1}{1+x^2}$

$$\boxed{\text{Why is } D \arctan x = \frac{1}{1+x^2} \text{ ?}}$$

PROOF : $y = \arctan x \implies x = \tan y$ (Definition of inverse tangent)

$\implies 1 = \sec^2 y \cdot y'$ (Implicit differentiation)

$\implies y' = \dfrac{1}{\sec^2 y}$ (Solve for y.)

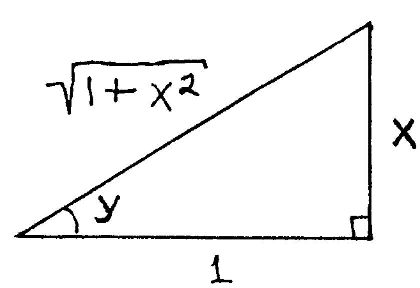
$\implies y' = \dfrac{1}{1/\cos^2 y}$ (Definition of secant)

$\implies y' = \cos^2 y$

$\implies y' = (\cos y)^2$

$\implies y' = \left(\dfrac{1}{\sqrt{1+x^2}}\right)^2$ (Definition of cosine and Pythagorean Theorem from right triangle)

$\implies y' = \dfrac{1}{1+x^2}$



$\tan y = x = \dfrac{x}{1}$

Arctangent Antiderivatives

Rule: $\int \frac{1}{1+x^2}\,dx = \text{arctangent}\, x + C$

Rule: $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\,\text{arctangent}\left(\frac{x}{a}\right) + C$

WHY? $\int \frac{1}{a^2+x^2}\,dx = \int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx$

$= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$ (Let $u = \frac{x}{a} \xrightarrow{D}$
$du = \frac{1}{a}\,dx \rightarrow$
$a\,du = dx$)

$= \frac{1}{a^2}\int \frac{a}{1+u^2}\,du$

$= \frac{1}{a}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{a}\arctan u + C = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

Ex: $\int \frac{1}{16+x^2}\,dx = \int \frac{1}{4^2+x^2}\,dx = \frac{1}{4}\,\text{arctangent}\left(\frac{x}{4}\right) + C$

Ex: $\int \frac{1}{x^2+4x+13}\,dx = \int \frac{1}{(x^2+4x+4)+9}\,dx$

$= \int \frac{1}{(x+2)^2+3^2}\,dx$ (Let $u = x+2 \rightarrow \cdots$)

$= \frac{1}{3}\arctan\left(\frac{x+2}{3}\right) + C$