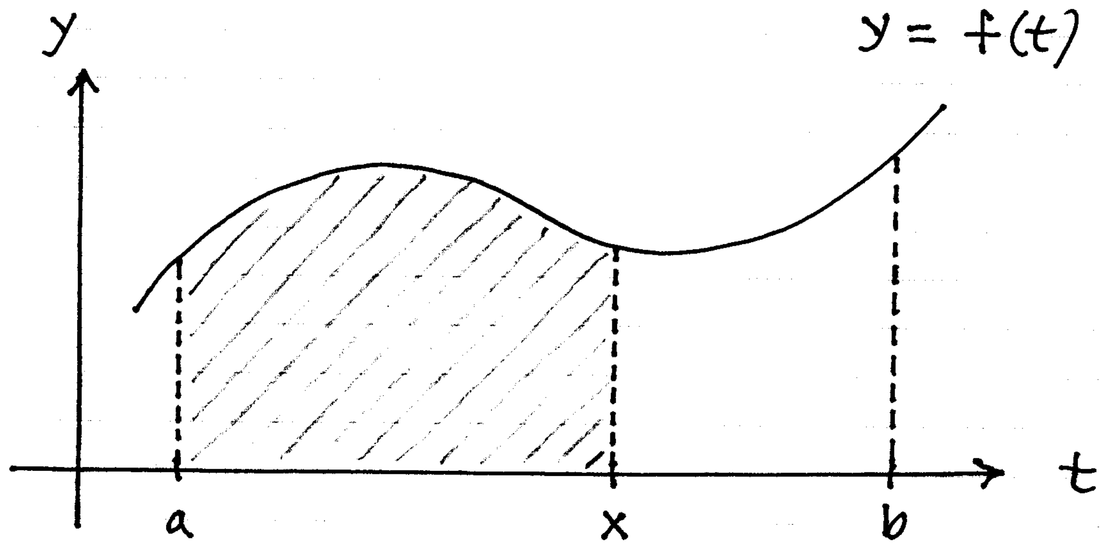
Lecture 3

Fundamental Theorems of Calculus

We seek a shortcut for evaluating the definite integral $\int_a^b f(x)\,dx$?

Let function $\boxed{F(x) = \int_a^x f(t)\,dt}$ be the area of the shaded region for $a \le t \le x$.

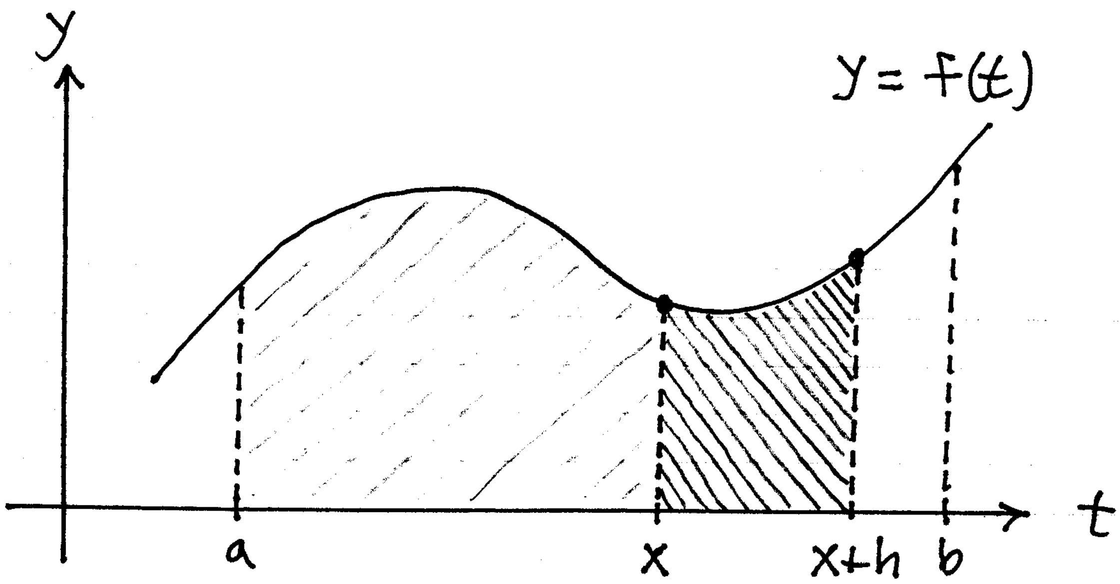


Note: i.) $F(a) = \int_a^a f(t)\,dt = 0$

ii.) $F(b) = \int_a^b f(t)\,dt$

Differentiate function F :

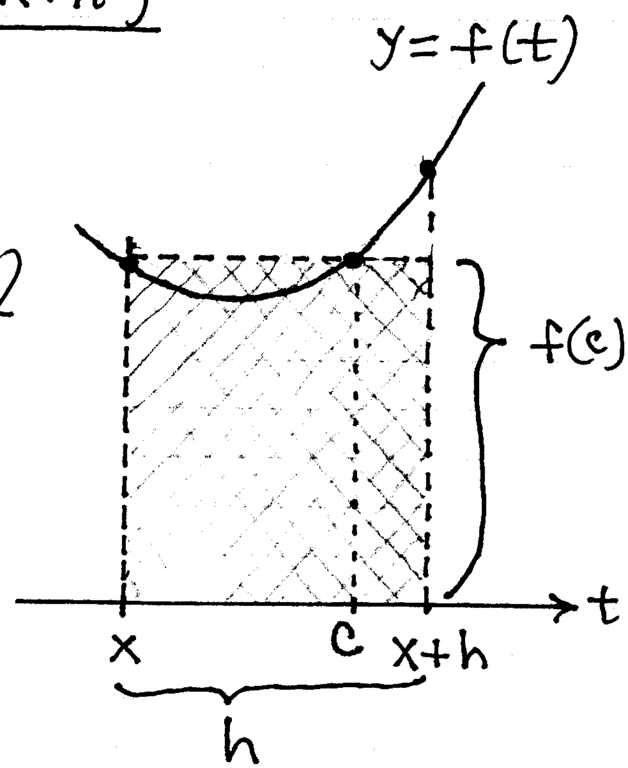
$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$



$$= \lim_{h\to 0} \frac{(\text{Area } a \le t \le x+h) - (\text{Area } a \le t \le x)}{h}$$

$$= \lim_{h\to 0} \frac{(\text{Area } x \le t \le x+h)}{h}$$

(by MVT for integrals)



$$= \lim_{h\to 0} \frac{(\text{Area rectangle})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h} \cdot f(c)}{\cancel{h}}$$

$$= \lim_{h\to 0} f(c)$$

$$= f(x),$$

i.e., $\boxed{F'(x) = f(x)}$

so that $F(x)$ is an antiderivative of $f(x)$. Now consider <u>any antiderivative</u> $A(x)$ of $f(x)$. We know that

$$F(x) = A(x) + C .$$

<u>Here is the shortcut</u>:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

$$= F(b)$$

$$= F(b) - 0$$

$$= F(b) - F(a)$$

$$= (A(b) + \cancel{C}) - (A(a) + \cancel{C})$$

$$= A(b) - A(a) \text{ , i.e.,}$$

$$\boxed{\int_a^b f(x)\,dx = A(x)\Big|_a^b = A(b) - A(a)}$$

<u>FTC 1</u>: $D \int_a^x f(t)\,dt = f(x)$

and $D \int_a^{g(x)} f(t)\,dt = f(g(x)) \cdot g'(x)$

<u>FTC 2</u>: $\int_a^b f(x)\,dx = A(x)\Big|_a^b = A(b) - A(a)$