

RECALL: If  $L$  is a Leslie matrix and  $N(0)$  is the initial age distribution, then

$$N(1) = L N(0),$$

$$N(2) = L N(1) = L(L N(0)) = L^2 N(0),$$

$$N(3) = L N(2) = L(L^2 N(0)) = L^3 N(0), \dots$$

and  $\boxed{N(k) = L^k N(0)}$  for  $k=1, 2, 3, \dots$ .

NOTE: If  $V$  is an eigenvector for  $L$  with eigenvalue  $\lambda$ , then

$$LV = \lambda V,$$

$$L^2 V = L(LV) = L(\lambda V) = \lambda(LV) = \lambda(\lambda V) = \lambda^2 V,$$

$$L^3 V = L(L^2 V) = L(\lambda^2 V) = \lambda^2(LV) = \lambda^2(\lambda V) = \lambda^3 V, \dots$$

and  $\boxed{L^k V = \lambda^k V}$  for  $k=1, 2, 3, \dots$ .

The following example illustrates the eigenvalue properties for a Leslie matrix  $L$ .

Example:  $L = \begin{bmatrix} 3 & 8 \\ 1/2 & 0 \end{bmatrix} \rightarrow$

$$\det(L - \lambda I) = \det \begin{bmatrix} 3-\lambda & 8 \\ 1/2 & 0-\lambda \end{bmatrix} = 0 \rightarrow$$

$$(3-\lambda)(-\lambda) - 8\left(\frac{1}{2}\right) = 0 \rightarrow$$

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \rightarrow$$

$\boxed{\lambda = 4}$ ,  $\boxed{\lambda = -1}$ ; now solve

$(A - \lambda I)X = 0$  for each eigenvalue:

$\boxed{\lambda = 4}$ :  $\left[ \begin{array}{cc|c} -1 & 8 & 0 \\ 1/2 & -4 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} -1 & 8 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow$

$$-x_1 + 8x_2 = 0, \text{ so let } x_2 = t \text{ any } \# \rightarrow$$

$$x_1 = 8x_2 = 8t; \text{ then}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8t \\ t \end{bmatrix} = t \begin{bmatrix} 8 \\ 1 \end{bmatrix}, \text{ so choose}$$

$$\text{eigenvector } V_1 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

$\boxed{\lambda = -1}$ :  $\left[ \begin{array}{cc|c} 4 & 8 & 0 \\ 1/2 & 1 & 0 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow$

$$x_1 + 2x_2 = 0, \text{ so let } x_2 = t \text{ any } \# \rightarrow$$

$x_1 = -2x_2 = -2t$  ; then

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ , so choose eigenvector  $V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ .

Let's assume that an initial age distribution is  $N(0) = \begin{bmatrix} 66 \\ 17 \end{bmatrix}$  and write  $\begin{bmatrix} 66 \\ 17 \end{bmatrix}$  as a linear combination of eigenvectors  $V_1$  and  $V_2$  :

$$\begin{bmatrix} 66 \\ 17 \end{bmatrix} = c_1 \begin{bmatrix} 8 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow$$

$$\begin{cases} 66 = 8c_1 - 2c_2 \\ 17 = c_1 + c_2 \end{cases}$$

$$17 = c_1 + c_2 \rightarrow 34 = 2c_1 + 2c_2$$

$$\rightarrow 100 = 10c_1 \rightarrow c_1 = 10 \text{ and } c_2 = 7.$$

We now have

$$N(0) = \begin{bmatrix} 66 \\ 17 \end{bmatrix} = (10) \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (7) \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Let's use the fact that

$$L \begin{bmatrix} 8 \\ 1 \end{bmatrix} = (4) \begin{bmatrix} 8 \\ 1 \end{bmatrix} \text{ and } L \begin{bmatrix} -2 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

to determine the age distribution when  $t=8$ , i.e., find  $N(8)$ :

$$N(8) = L^8 N(0)$$

$$= L^8 \begin{bmatrix} 66 \\ 17 \end{bmatrix}$$

$$= L^8 \left\{ (10) \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (7) \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

$$= (10) L^8 \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (7) L^8 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= (10) (4)^8 \begin{bmatrix} 8 \\ 1 \end{bmatrix} + (7) (-1)^8 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 80 (4)^8 - 14 \\ 10 (4)^8 + 7 \end{bmatrix}$$

$$= \begin{bmatrix} 5,242,866 \\ 655,367 \end{bmatrix} .$$

The % of 0-year olds is

$$\frac{5,242,866}{5,242,866 + 655,367} \approx 88.9\% ;$$

the % of 1-year olds is

$$\frac{655,367}{5,242,866 + 655,367} \approx 11.1\% .$$

CONCLUSION: The STABLE AGE DISTRIBUTION is  $\begin{bmatrix} 8 \\ 1 \end{bmatrix}$ , so for large values of  $t$  we can expect about  $\frac{8}{8+1} = \frac{8}{9} \approx 88.9\%$  0-yr. olds and  $\frac{1}{8+1} = \frac{1}{9} \approx 11.1\%$  1-yr. olds .

Note also that  $\lambda_1 = 4 > 0$ , so the population is increasing in size .