Math 17B
Kouba
Mixture Problems (Solutions)

Example 1: $\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$

$= \left(\frac{2 \text{ lbs.}}{\text{gal.}}\right)\left(\frac{3 \text{ gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lbs.}}{500 \text{ gal.}}\right)\left(\frac{3 \text{ gal.}}{\text{min.}}\right) \rightarrow$

a.) $\frac{dS}{dt} = 6 - \frac{3}{500}S$ and $S(0) = 25$ lbs.

(Separable D.E.) $\rightarrow$

$\frac{1}{6 - \frac{3}{500}S}\, dS = dt \rightarrow$

$\int \frac{1}{6 - \frac{3}{500}S} \cdot \frac{500}{500}\, dS = \int 1\, dt \rightarrow$

$\int \frac{500}{3000 - 3S}\, dS = t + C \rightarrow$

$500 \cdot \frac{-1}{3} \ln(3000 - 3S) = t + C$;

and $t = 0$, $S = 25 \rightarrow -\frac{500}{3} \ln 2925 = C \rightarrow$

$-\frac{500}{3} \ln(3000 - 3S) = t - \frac{500}{3} \ln 2925 \rightarrow$

(Solve for S.)

$\ln(3000 - 3S) = \frac{-3}{500}t + \ln 2925 \rightarrow$

$e^{\ln(3000 - 3S)} = e^{\frac{-3}{500}t + \ln 2925} \rightarrow$

$$3000 - 3S = e^{\ln 2925}\, e^{\frac{-3}{500}t} \rightarrow$$

$$3000 - 3S = 2925\, e^{\frac{-3}{500}t} \rightarrow$$

$$3S = 3000 - 2925\, e^{\frac{-3}{500}t} \rightarrow$$

$$\boxed{S = 1000 - 975\, e^{\frac{-3}{500}t}}$$

b.) $t = 10$ min. $\rightarrow S = 1000 - 975\, e^{\frac{-3}{500}(10)}$

$\rightarrow S \approx 81.78$ lbs.

$t = 60$ min. $\rightarrow S = 1000 - 975 \cdot e^{\frac{-3}{500}(60)}$

$\rightarrow S \approx 319.77$ lbs.

c.) $\lim_{t \to \infty} S = \lim_{t \to \infty} \left(1000 - 975 \cdot e^{\frac{-3}{500}t}\right)$

$= 1000 - 975 \cdot e^{-\infty}$ ($e^{-\infty} \to 0$)

$= 1000$ lbs.

Example 2:

a.) $t = 1$ min.: $500 - 2(1) = 498$ gal.,
$t = 10$ min.: $500 - 2(10) = 480$ gal.,
$t = 50$ min.: $500 - 2(50) = 400$ gal.,
t min.: $500 - 2t$ gal.;

$500 - 2t = 0 \rightarrow 500 = 2t \rightarrow$
$t = 250$ min. (empty)

b.) $\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$

$= \left(\frac{2 \text{ lbs.}}{\text{gal.}}\right)\left(\frac{3 \text{ gal.}}{\text{min.}}\right) - \left(\frac{S \text{ lbs.}}{500 - 2t \text{ gal.}}\right)\left(\frac{5 \text{ gal.}}{\text{min.}}\right) \rightarrow$

$\frac{dS}{dt} = 6 - \frac{5S}{500 - 2t}$ and $S(0) = 25$ lbs.

(First-order linear D.E.) $\rightarrow$

$\frac{dS}{dt} + \frac{5}{500 - 2t} \cdot S = 6$

$\left(\mu = e^{\int \frac{5}{500-2t}\, dt} = e^{-\frac{5}{2}\ln(500-2t)}\right.$

$= e^{\ln(500-2t)^{-5/2}} = \left.(500-2t)^{-5/2}\right) \rightarrow$

$$(500-2t)^{-5/2}\cdot\frac{dS}{dt}+(500-2t)^{-5/2}\cdot\frac{5}{500-2t}S=6(500-2t)^{-5/2}\rightarrow$$

$$(500-2t)^{-5/2}\cdot\frac{dS}{dt}+5(500-2t)^{-7/2}\cdot S=6(500-2t)^{-5/2}\rightarrow$$

$$D\left[(500-2t)^{-5/2}\cdot S\right]=6(500-2t)^{-5/2}\rightarrow$$

$$(500-2t)^{-5/2}\cdot S=\int 6(500-2t)^{-5/2}dt\rightarrow$$

$$(500-2t)^{-5/2}\cdot S=6\cdot\frac{-1}{2}\cdot\frac{-2}{3}(500-2t)^{-3/2}+c\rightarrow$$

$$(500-2t)^{-5/2}\cdot S=2(500-2t)^{-3/2}+c\rightarrow$$

$$S=2(500-2t)+(500-2t)^{5/2}\cdot c\ ;$$

and $t=0$, $S=25\rightarrow$

$$25=1000+500^{5/2}\cdot c\rightarrow$$

$$c=\frac{-975}{500^{5/2}}\quad \text{then}$$

$$\boxed{S=1000-4t-\frac{975}{500^{5/2}}(500-2t)^{5/2}}$$

c.) $t = 10$ min.: $S \approx 79.6$ lbs.,
$t = 100$ min.: $S \approx 328.1$ lbs.,
$t = 200$ min.: $S \approx 182.6$ lbs.,
$t = 240$ min.: $S \approx 39.7$ lbs.

d.) Find maximum S:

$$S = 1000 - 4t - \frac{975}{500^{5/2}}(500-2t)^{5/2} \quad \underline{D} \rightarrow$$

$$\frac{dS}{dt} = -4 - \frac{975}{500^{5/2}} \cdot \frac{5}{2}(500-2t)^{3/2} \cdot (-2)$$

$$= -4 + \frac{4875}{500^{5/2}}(500-2t)^{3/2} = 0 \rightarrow$$

$$\frac{4875}{500^{5/2}}(500-2t)^{3/2} = 4 \rightarrow$$

$$(500-2t)^{3/2} = \frac{4(500^{5/2})}{4875} \rightarrow$$

$$500 - 2t = \left[\frac{4 \cdot 500^{5/2}}{4875}\right]^{2/3} \rightarrow$$

$$2t = 500 - \left[\frac{4 \cdot 500^{5/2}}{4875}\right]^{2/3} \rightarrow$$

$$t = 250 - \frac{1}{2}\left[\frac{4 \cdot 500^{5/2}}{4875}\right]^{2/3} \approx 112 \text{ min.}$$

and $S \approx 331.3$ lbs.