RECALL: If L is a Leslie matrix and $N(0)$ is the initial age distribution, then

$$N(1) = L\,N(0),$$
$$N(2) = L\,N(1) = L(L\,N(0)) = L^2 N(0),$$
$$N(3) = L\,N(2) = L(L^2 N(0)) = L^3 N(0), \ldots$$

and $\boxed{N(k) = L^k N(0)}$ for $k = 1, 2, 3, \ldots$.

NOTE: If V is an eigenvector for L with eigenvalue λ, then

$$LV = \lambda V,$$
$$L^2 V = L(LV) = L(\lambda V) = \lambda(LV) = \lambda(\lambda V) = \lambda^2 V,$$
$$L^3 V = L(L^2 V) = L(\lambda^2 V) = \lambda^2 (LV) = \lambda^2(\lambda V) = \lambda^3 V, \ldots$$

and $\boxed{L^k V = \lambda^k V}$ for $k = 1, 2, 3, \ldots$.

The following example illustrates the eigenvalue properties for a Leslie matrix L.

<u>Example</u>: $L = \begin{bmatrix} 3 & 8 \\ 1/2 & 0 \end{bmatrix} \rightarrow$

$\det(L - \lambda I) = \det \begin{bmatrix} 3-\lambda & 8 \\ 1/2 & 0-\lambda \end{bmatrix} = 0 \rightarrow$

$(3-\lambda)(-\lambda) - 8\left(\frac{1}{2}\right) = 0 \longrightarrow$

$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0 \longrightarrow$

$\boxed{\lambda = 4}$, $\boxed{\lambda = -1}$; now solve

$(A - \lambda I)X = 0$ for each eigenvalue:

$\boxed{\lambda = 4}$: $\left[\begin{array}{cc|c} -1 & 8 & 0 \\ 1/2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 8 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$

$-x_1 + 8x_2 = 0$, so let $x_2 = t$ any # $\rightarrow$

$x_1 = 8x_2 = 8t$; then

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8t \\ t \end{bmatrix} = t \begin{bmatrix} 8 \\ 1 \end{bmatrix}$, so choose

eigenvector $V_1 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$.

$\boxed{\lambda = -1}$: $\left[\begin{array}{cc|c} 4 & 8 & 0 \\ 1/2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$

$x_1 + 2x_2 = 0$, so let $x_2 = t$ any # $\rightarrow$

$x_1 = -2x_2 = -2t$; then

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, so choose

eigenvector $V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Let's assume that an initial age distribution is $N(0) = \begin{bmatrix} 66 \\ 17 \end{bmatrix}$ and write $\begin{bmatrix} 66 \\ 17 \end{bmatrix}$ as a linear combination of eigenvectors V_1 and V_2 :

$$\begin{bmatrix} 66 \\ 17 \end{bmatrix} = c_1 \begin{bmatrix} 8 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow$$

$$\begin{cases} 66 = 8c_1 - 2c_2 \\ 17 = c_1 + c_2 \rightarrow 34 = 2c_1 + 2c_2 \end{cases}$$

$\rightarrow 100 = 10c_1 \rightarrow c_1 = 10$ and $c_2 = 7$.

We now have

$$N(0) = \begin{bmatrix} 66 \\ 17 \end{bmatrix} = (10)\begin{bmatrix} 8 \\ 1 \end{bmatrix} + (7)\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Let's use the fact that

$$L\begin{bmatrix}8\\1\end{bmatrix} = (4)\begin{bmatrix}8\\1\end{bmatrix} \text{ and } L\begin{bmatrix}-2\\1\end{bmatrix} = (-1)\begin{bmatrix}-2\\1\end{bmatrix}$$

to determine the age distribution when $t=8$, i.e., find $N(8)$:

$$\begin{aligned}
N(8) &= L^8 N(0) \\
&= L^8 \begin{bmatrix}66\\17\end{bmatrix} \\
&= L^8 \left\{ (10)\begin{bmatrix}8\\1\end{bmatrix} + (7)\begin{bmatrix}-2\\1\end{bmatrix} \right\} \\
&= (10) L^8 \begin{bmatrix}8\\1\end{bmatrix} + (7) L^8 \begin{bmatrix}-2\\1\end{bmatrix} \\
&= (10)(4)^8 \begin{bmatrix}8\\1\end{bmatrix} + (7)(-1)^8 \begin{bmatrix}-2\\1\end{bmatrix} \\
&= \begin{bmatrix}80(4)^8 - 14\\10(4)^8 + 7\end{bmatrix} \\
&= \begin{bmatrix}5,242,866\\655,367\end{bmatrix}.
\end{aligned}$$

The % of 0-year olds is

$$\frac{5{,}242{,}866}{5{,}242{,}866 + 655{,}367} \approx 88.9\% \; ;$$

the % of 1-year olds is

$$\frac{655{,}367}{5{,}242{,}866 + 655{,}367} \approx 11.1\% \; .$$

<u>CONCLUSION</u>: The STABLE AGE DISTRIBUTION is $\begin{bmatrix} 8 \\ 1 \end{bmatrix}$, so for large values of t we can expect about

$$\frac{8}{8+1} = \frac{8}{9} \approx 88.9\% \text{ 0-yr. olds}$$

and

$$\frac{1}{8+1} = \frac{1}{9} \approx 11.1\% \text{ 1-yr. olds}.$$

Note also that $\lambda_1 = 4 > 1$, so the population is increasing in size.