Math 17B
Kouba
Exam 2

Your Name : KEY

Your EXAM ID Number __________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 8 pages, including the cover page.

6. You will be graded on proper use of integral and derivative notation.

7. You will be graded on proper use of limit notation.

8. You have until 10:50 a.m. to finish the exam.

1.) (9 pts. each) Solve the following differential equations. (Let $u = x$, $dv = e^x dx$

a.) $\dfrac{dy}{dx} = xe^x \rightarrow y = \int xe^x\,dx \rightarrow du = dx$, $v = e^x$)

$= xe^x - \int e^x\,dx = xe^x - e^x + c$

b.) $\dfrac{dy}{dx} = \sec y \cdot \sec^2 x \longrightarrow \int \dfrac{1}{\sec y}\,dy = \int \sec^2 x\,dx \rightarrow$

$\int \cos y\,dy = \tan x + C \rightarrow \sin y = \tan x + C$

c.) $x^2 \dfrac{dy}{dx} + xy = x^3 + 1 \longrightarrow \dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = \dfrac{x^3+1}{x^2}$

$\left(\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x\right) \rightarrow x \cdot \dfrac{dy}{dx} + y = \dfrac{x^3+1}{x} \rightarrow$

$D[xy] = x^2 + \dfrac{1}{x} \rightarrow xy = \int \left(x^2 + \frac{1}{x}\right)dx \rightarrow$

$xy = \frac{1}{3}x^3 + \ln|x| + c$

2.) (9 pts.) Compute T_5, the Trapezoidal Estimate with $n = 5$, for $\int_0^5 \underbrace{\sqrt{x+2}}_{f(x)}\, dx$.

0 1 2 3 4 5

, $h = \dfrac{5-0}{5} = 1$

$T_5 = \dfrac{h}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right]$

$= \dfrac{1}{2}\left[\sqrt{2} + 2\sqrt{3} + 2\sqrt{4} + 2\sqrt{5} + 2\sqrt{6} + \sqrt{7}\right]$

≈ 10.45

3.) (5 pts. each) Determine the equilibria for $\frac{dN}{dt} = 5N - N^2$ and determine their stability using $g(N) = 5N - N^2 = N(5-N) = 0 \rightarrow N=0,\ N=5$

a.) the graphical approach (sign chart).

$$\begin{array}{ccccc} - & 0 & + & 0 & - \\ & N=0 & & N=5 & \\ & \text{(unstable)} & & \text{(Stable)} & \end{array} \qquad g(N)$$

b.) the analytical approach (eigenvalue method). $g(N) = 5N - N^2 \xrightarrow{D}$

$g'(N) = 5 - 2N$;

$g(0) = 5 - 2(0) = 5 > 0$ so $N=0$ $\underline{\text{unstable}}$;

$g(5) = 5 - 2(5) = -5 < 0$ so $N=5$ $\underline{\text{Stable}}$

4.) (10 pts.) What should n be in order that the Midpoint Estimate estimate the exact value of $\int_0^2 (x+3)^{-2}\, dx$ with absolute error at most 0.001 ? The absolute error for the Midpoint Estimate is given by $|E_n| \le (b-a)\frac{h^2}{24}\left\{ \max_{a \le x \le b} |f''(x)| \right\}$.

$f(x) = (x+3)^{-2} \xrightarrow{D} f'(x) = -2(x+3)^{-3} \xrightarrow{D} f''(x) = 6(x+3)^{-4}$

$\rightarrow f''(x) = \frac{6}{(x+3)^4}$; $\max_{0 \le x \le 2} |f''(x)| = \max_{0 \le x \le 2} \frac{6}{(x+3)^4} = \frac{6}{(0+3)^4} = \frac{2}{27}$;

$h = \frac{2-0}{n} = \frac{2}{n}$; then error

$|E_n| \le (2-0) \frac{\left(\frac{2}{n}\right)^2}{24} \cdot \left\{\frac{2}{27}\right\} = \frac{\cancel{16}^{\,2}}{\cancel{24}_{\,3}} \cdot \frac{1}{27} \cdot \frac{1}{n^2} = \underline{\frac{2}{81 n^2} \le 0.001}$

$\rightarrow n^2 \ge \frac{2}{81(0.001)} \rightarrow n \ge \sqrt{\frac{2}{81(0.001)}} \approx 4.97$

so choose $\boxed{n \ge 5}$.

5.) (8 pts. each) Evaluate the following improper integrals.

a.) $\int_{\sqrt{3}}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{A\to\infty} \int_{\sqrt{3}}^{A} \frac{1}{1+x^2}\,dx$

$= \lim_{A\to\infty} \arctan x \Big|_{\sqrt{3}}^{A}$

$= \lim_{A\to\infty} (\arctan A - \arctan\sqrt{3})$

$= \arctan\infty - \frac{\pi}{3}$

$= \frac{\pi}{2} - \frac{\pi}{3}$

$= \frac{\pi}{6}$

b.) $\int_{0}^{1} \frac{1}{\sqrt{1-x}}\,dx = \lim_{A\to 1^-} \int_{0}^{A} (1-x)^{-1/2}\,dx$

$= \lim_{A\to 1^-} -2(1-x)^{1/2} \Big|_{0}^{A}$

$= \lim_{A\to 1^-} [-2(1-A)^{1/2} - -2(1)^{1/2}]$

$= -2(0) + 2$

$= 2$

6.) (9 pts.) Determine $P_2(x)$, the second-degree Taylor Polynomial centered at $x = 0$, for $f(x) = \dfrac{x}{x-1}$.

$$\xrightarrow{D} f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} = \frac{-1}{(x-1)^2} = -(x-1)^{-2}$$

$$\xrightarrow{D} f''(x) = 2(x-1)^{-3} = \frac{2}{(x-1)^3} \quad ; \quad a_n = \frac{f^{(n)}(0)}{n!} \longrightarrow$$

$$a_0 = \frac{f(0)}{0!} = \frac{0}{1} = 0,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{-1}{1} = -1,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-2}{2} = -1$$

$$\rightarrow P_2(x) = a_0 + a_1(x-0) + a_2(x-0)^2 \rightarrow$$

$$P_2(x) = -x - x^2$$

7.) (9 pts.) (Mixture Problem) Let S be the amount (pounds) of sugar in a tank at time t (minutes). A solution containing 2 pounds of sugar per gallon begins flowing into the tank at the rate of 5 gallons per minute and the well-stirred mixture flows out of the tank at the rate of 3 gallons per minute. Initially, the tank holds 100 gallons with 20 pounds of sugar. SET UP BUT DO NOT SOLVE a differential equation with initial conditions for the rate $\dfrac{dS}{dt}$.

$$\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= \left(\frac{2\text{ lbs.}}{\text{gal.}}\right)\left(\frac{5\text{ gal.}}{\text{min.}}\right) - \left(\frac{S\text{ lbs.}}{100+2t\text{ gal.}}\right)\left(\frac{3\text{ gal.}}{\text{min.}}\right)$$

and $S(0) = 20$ lbs.

8.) (**10**pts.) The following data are plotted on a semi-log graph on the following page. Use the graph to solve for N and determine the growth rate $\frac{dN}{dt}$ in autonomous form.

t	N
0	7
1.5	19.8
3	56
4	112
5.5	316.8
7	896

assume

$\log N = mt + \log C \rightarrow$

$\log N = mt + \log 7$ and $t=3,\ N=56 \rightarrow$

$\log 56 = 3m + \log 7 \rightarrow 3m = \log 56 - \log 7 \rightarrow$

$3m = \log\left(\frac{56}{7}\right) \rightarrow m = \frac{1}{3}\log 8 \rightarrow m = \log 8^{1/3} \rightarrow$

$m = \log 2 \rightarrow \underline{\log N = t \cdot \log 2 + \log 7}$;

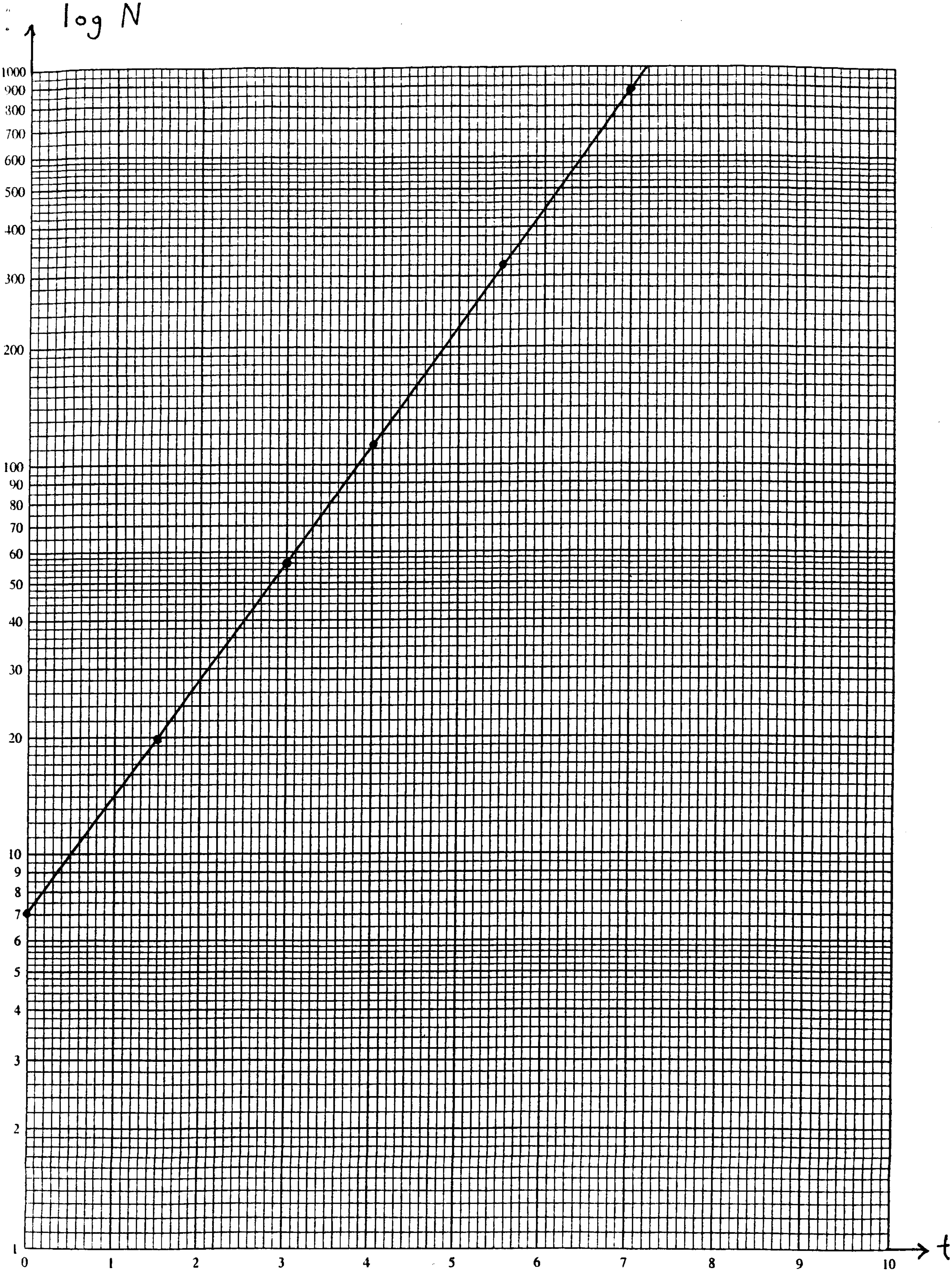
solve for N :

$10^{\log N} = 10^{t \cdot \log 2 + \log 7} = 10^{\log 2^t} \cdot 10^{\log 7} \rightarrow$

$N = 2^t \cdot 7 \rightarrow \boxed{N = 7 \cdot 2^t} \xrightarrow{D}$

$\frac{dN}{dt} = 7 \cdot 2^t \cdot \ln 2 = N \cdot \ln 2 \rightarrow$

$$\boxed{\frac{dN}{dt} = \ln 2 \cdot N}$$


log N
1000
900
800
700
600
500
400
300
200
100
90
80
70
60
50
40
30
20
10
9
8
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
t

The following EXTRA CREDIT problem is OPTIONAL. It is worth 10 points.

1.) Solve the following differential equation. ~~Solve explicitly for y.~~

$$(1+e^x)\frac{dy}{dx} - \cot^2 y = 0 \longrightarrow$$

$$(1+e^x)\frac{dy}{dx} = \cot^2 y \rightarrow$$

$$\int \frac{1}{\cot^2 y}\,dy = \int \frac{1}{1+e^x}\,dx \longrightarrow$$

$$\int \tan^2 y\,dy = \int \frac{1}{1+e^x}\cdot\frac{e^{-x}}{e^{-x}}\,dx \longrightarrow$$

$$\int (\sec^2 y - 1)\,dy = \int \frac{e^{-x}}{e^{-x}+1}\,dx \longrightarrow$$

$$\tan y - y = -\ln(e^{-x}+1) + c$$