Math 17B
Kouba
Exam 3 (Practice)

Your Name : KEY

Your EXAM ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 7 pages, including the cover page.

6. You have until 10:50 a.m. to finish the exam.

1.) Consider the Leslie matrix $L = \begin{pmatrix} 0 & 2.3 & 1.8 & 1.2 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{pmatrix}$

a.) (2 pts.) How many age classes are in this population ? 4

b.) (2 pts.) What percentage of 1-year old females survive to the end of the following breeding season ? 40%

c.) (2 pts.) What percenatge of 2-year old females survive to the end of the following breeding season ? 60%

d.) (2 pts.) What is an average number of female offspring for a 0-year old female ? 0

e.) (2 pts.) What is an average number of female offspring for a 3-year old female ? 1.2

f.) (6 pts.) If $N(0) = \begin{pmatrix} 30 \\ 10 \\ 20 \\ 40 \end{pmatrix}$, determine $N(1)$. How many 2-year old females will there be at the end of breeding season 1 ?

$$N(1) = \begin{bmatrix} 0 & 2.3 & 1.8 & 1.2 \\ 0.1 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 10 \\ 20 \\ 40 \end{bmatrix} = \begin{bmatrix} 23+36+48 \\ 3 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 107 \\ 3 \\ 4 \\ 12 \end{bmatrix}$$

There are 4 2-yr. olds.

2.) (8 pts.) GOOP weighs 50 lbs./ft.^{3} and SLUDGE weighs 70 lbs./ft.^{3} How many cubic feet of each material should be mixed together in order to result in 10 ft.^{3} of mixture weighing 65 lbs./ft.^{3} ?

x: ft.^{3} of GOOP, y: ft.^{3} of SLUDGE then

$$\left.\begin{matrix} x+y=10 \\ 50x+70y=65(10) \end{matrix}\right\} \rightarrow \left.\begin{matrix} y=10-x \\ 50x+70(10-x)=650 \end{matrix}\right\} \rightarrow$$

$$50x+700-70x=650 \rightarrow 50=20x \rightarrow$$

$$x=2.5 \text{ ft.}^3, \quad y=7.5 \text{ ft.}^3$$

3.) (8 pts. each) Use matrix reduction to solve each of the following systems of equations.

a.) $\begin{cases} x - y = 1 \\ y + 3z = -1 \\ 2x + y - z = 9 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 3 & -1 \\ 2 & 1 & -1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 3 & -1 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -10 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow$$

$X = 3, \ Y = 2, \ Z = -1$

b.) $\begin{cases} x + y = 2 \\ y - z = 0 \\ 2x + 2z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \rightarrow \quad 0 = -1 \ ! \ \text{so}$$

no solution

4.) Determine the inverse for each matrix.

a.) (6 pts.) $A = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 0 & 2 & -2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 3/2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 3/2 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 0 & 1/2 \\ -1 & 3/2 \end{bmatrix}$$

b.) (10 pts.) $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \rightarrow A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}$$

5.) (10 pts.) The given 2x2 matrix is a Leslie matrix. Determine a stable age distribution for this matrix. Is the population increasing or decreasing in size ?

$A = \begin{pmatrix} 2 & 4 \\ 3/4 & 0 \end{pmatrix}$ $\det(A-\lambda I) = \det\begin{bmatrix} 2-\lambda & 4 \\ 3/4 & -\lambda \end{bmatrix}$

$= (2-\lambda)(-\lambda) - (4)(3/4) = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$

$\rightarrow \boxed{\lambda_1 = 3}$, $\lambda_2 = -1$; since $\lambda_1 = 3 > 1$, the population is increasing ;

For $\lambda_1 = 3$: solve $(A-\lambda I)X = 0 \rightarrow$

$\left[\begin{array}{cc|c} -1 & 4 & 0 \\ 3/4 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow -x_1 + 4x_2 = 0$ so

let $x_2 = t$ any # $\rightarrow x_1 = 4x_2 = 4t$ so

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, then $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is stable age distribution

6.) (10 pts.) Find eigenvalues and the corresponding eigenvectors for the matrix

$A = \begin{pmatrix} -2 & 5 \\ 4 & 6 \end{pmatrix}$. $\det(A-\lambda I) = \det\begin{bmatrix} -2-\lambda & 5 \\ 4 & 6-\lambda \end{bmatrix}$

$= (-2-\lambda)(6-\lambda) - (5)(4) = \lambda^2 - 6\lambda + 2\lambda - 12 - 20$

$= \lambda^2 - 4\lambda - 32 = (\lambda-8)(\lambda+4) = 0 \rightarrow \lambda = 8, \lambda = -4$.

For $\lambda_1 = 8$: $\left[\begin{array}{cc|c} -10 & 5 & 0 \\ 4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$

$x_1 - \frac{1}{2}x_2 = 0$ so let $x_2 = t$ any # $\rightarrow x_1 = \frac{1}{2}x_2 = \frac{1}{2}t$,

then $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = \frac{1}{2}t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so $\boxed{V_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ is eigenvector.

For $\lambda_2 = -4$: $\left[\begin{array}{cc|c} 2 & 5 & 0 \\ 4 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow 2x_1 + 5x_2 = 0$ so

let $x_2 = t$ any # $\rightarrow 2x_1 = -5x_2 = -5t \rightarrow x_1 = \frac{-5}{2}t$, then

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2}t \\ t \end{bmatrix} = \frac{1}{2}t\begin{bmatrix} -5 \\ 2 \end{bmatrix}$, so $\boxed{V_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}}$ is eigenvector.

7.) (7 pts.) Determine if the following statement is TRUE or FALSE : If matrix A has 4 entries in it and matrix B has 4 entries in it and matrix AB is defined, then matrix AB must have 4 entries in it. If this statement is TRUE, briefly explain why. If this statemnet is FALSE, provide an example contradicting the statement.

FALSE :

Let $A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ (each have 4 entries)

4×1 1×4

$$AB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$ (has 16 entries)

8.) (8 pts.) Give an example of a 2x2 rotation matrix which rotates vectors 60° clockwise.

rotation matrix : $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, for 60° clockwise let $\theta = -60° \rightarrow$

$$R = \begin{bmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{bmatrix} = \begin{bmatrix} 1/2 & -(-\sqrt{3}/2) \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \rightarrow$$

$$R = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$$

9.) (7 pts.) Assume that A is a 2x2 matrix. If $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, must it follow that $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$? If this statement is true, explain why. If this statement is false, find an example for which $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

False : Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but

$A \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The following EXTRA CREDIT problem is OPTIONAL. It is worth 10 points.

1.) Determine the 2x2 matrix which has the following eigenvalue/eigenvector combinations: $\lambda_1 = 2, V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = -3, V_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{cases} a+b=2 \\ c+d=2 \end{cases}$ and

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = (-3)\begin{bmatrix} -1 \\ 2 \end{bmatrix} \rightarrow \begin{cases} -a+2b=3 \\ -c+2d=-6 \end{cases}$, then

$\begin{cases} a+b=2 \\ -a+2b=3 \end{cases} \rightarrow 3b=5 \rightarrow b = 5/3$ and $a = 1/3$;

$\begin{cases} c+d=2 \\ -c+2d=-6 \end{cases} \rightarrow 3d=-4 \rightarrow d = \frac{-4}{3}$ and $c = \frac{10}{3} \rightarrow$

$A = \begin{bmatrix} 1/3 & 5/3 \\ 10/3 & -4/3 \end{bmatrix}$