Lecture 21

Math 17C
Kouba

Joint Probability Distributions

<u>Example</u>: Consider a large group of people made up of 5000 women and 5000 men. For each person record

i.) gender (M or F)

ii.) dominant hand (L or R).

Define random variable $X : \Omega \to \mathbb{R}$ by

$$X(x) = \begin{cases} 0 & \text{if } x \text{ is male (M)} \\ 1 & \text{if } x \text{ is female (F).} \end{cases}$$

Define random variable $Y : \Omega \to \mathbb{R}$ by

$$Y(y) = \begin{cases} 0 & \text{if } y \text{ is right-handed (R)} \\ 1 & \text{if } y \text{ is left-handed (L).} \end{cases}$$

Assume the results are as follows:

	(M) $X=0$	(F) $X=1$	Total
(R) $Y=0$	4710	4815	9525
(L) $Y=1$	290	185	475
Total	5000	5000	10,000

Convert this frequency table into a <u>joint probability</u> <u>distribution</u> by dividing all entries by 10,000 :

	(M) $X=0$	(F) $X=1$	Total
(R) $Y=0$	0.4710	0.4815	0.9525
(L) $Y=1$	0.0290	0.0185	0.0475
Total	0.5	0.5	1.0000

1.) What is the probability that a person selected at random is

a.) a left-handed male?

$$P(X=0, Y=1) = 0.0290$$

b.) a right-handed female?

$$P(X=1, Y=0) = 0.4815$$

c.) right-handed?

$$P(Y=0) = P(X=0, Y=0) + P(X=1, Y=0)$$
$$= 0.4710 + 0.4815 = 0.9525$$

2.) What is the probability that a person selected at random is female given that the person is left-handed?

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{0.0185}{0.0290+0.0185} \approx 0.3895$$

<u>Definition</u>: Let X and Y be random variables defined on sample space Ω. We say X and Y are <u>independent</u> if

$$P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$$

for all values x and all values y.

Example: Using the previous example, determine if X and Y are independent random variables:

$$P(X=0) = 0.4710 + 0.0290 = 0.5,$$
$$P(Y=1) = 0.0290 + 0.0185 = 0.0475,$$
$$P(X=0, Y=1) = 0.0290, \text{ then}$$
$$P(X=0)\cdot P(Y=1) = (0.5)(0.0475) = 0.02375$$
$$\neq 0.0290 = P(X=0, Y=1),$$

so variables X and Y are not independent.

Some Rules for Expected Value (Mean) and Variance

Let X and Y be discrete random variables defined on sample space Ω. Then

1.) $E(aX+b) = a\,E(X)+b$ for constants a and b.
2.) $\mathrm{var}(aX+b) = a^2\,\mathrm{var}(X)$ for constants a and b.
3.) $E(X+Y) = E(X) + E(Y)$
4.) (alternate) $\mathrm{var}(X) = E(X^2) - (E(X))^2$.

If random variables X and Y are independent, then

5.) $E(X\cdot Y) = E(X)\cdot E(Y)$
6.) $\mathrm{var}(X+Y) = \mathrm{var}(X) + \mathrm{var}(Y)$

Proof: 1.) $E(aX+b) = \sum_x (ax+b)\,p(x)$

$$= \sum_x (ax\,p(x) + b\,p(x))$$

$$= a \sum_x x\,p(x) + b \sum_x p(x)$$

$$= a\,E(X) + b(1) = a\,E(X) + b.$$

2.) Let $\mu = E(X)$. Then

$$\mathrm{var}(aX+b) = \sum_x \left((ax+b) - E(aX+b)\right)^2 p(x)$$

$$= \sum_x \left(ax+b-(a\mu+b)\right)^2 p(x) = \sum_x \left(a(x-\mu)\right)^2 p(x)$$

$$= a^2 \sum_x (x-\mu)^2 p(x) = a^2\, \mathrm{var}(X).$$

4.) $\mathrm{var}(X) = \sum_x (x-\mu)^2 p(x)$

$$= \sum_x (x^2 - 2\mu x + \mu^2)\, p(x)$$

$$= \sum_x x^2 p(x) - 2\mu \sum_x x\,p(x) + \mu^2 \sum_x p(x)$$

$$= E(X^2) - 2\mu(\mu) + \mu^2(1)$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - (E(X))^2$$