

Lecture 2

Math 17C

Laplace Transforms 2

Hw #2

Laplace Transforms (cont'd.)

Recall: $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$ so

$$\begin{aligned} 1.) \mathcal{L}\{\alpha f(t)\} &= \int_0^{\infty} e^{-st} \cdot \alpha f(t) dt \\ &= \alpha \int_0^{\infty} e^{-st} f(t) dt = \alpha \mathcal{L}\{f(t)\}; \end{aligned}$$

$$\begin{aligned} 2.) \mathcal{L}\{f(t) + g(t)\} &= \int_0^{\infty} e^{-st} (f(t) + g(t)) dt \\ &= \int_0^{\infty} (e^{-st} f(t) + e^{-st} g(t)) dt \\ &= \int_0^{\infty} e^{-st} f(t) dt + \int_0^{\infty} e^{-st} g(t) dt \\ &= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}. \end{aligned}$$

Ex: Use known formulas to find the Laplace transform of each function.

$$\begin{aligned} 1.) \mathcal{L}\{4+t\} &= \mathcal{L}\{4 \cdot 1\} + \mathcal{L}\{t\} \\ &= 4 \mathcal{L}\{1\} + \mathcal{L}\{t\} = 4 \cdot \frac{1}{s} + \frac{1}{s^2} \end{aligned}$$

$$2.) \mathcal{L}\{2e^t - 3e^{-2t}\}$$

$$= 2\mathcal{L}\{e^t\} - 3\mathcal{L}\{e^{-2t}\}$$

$$= 2 \cdot \frac{1}{s-1} - 3 \cdot \frac{1}{s-(-2)}$$

$$= \frac{2}{s-1} - \frac{3}{s+2}$$

$$3.) \mathcal{L}\{3 \cdot \sin 4t + \cos t\}$$

$$= 3\mathcal{L}\{\sin 4t\} + \mathcal{L}\{\cos t\}$$

$$= 3 \cdot \frac{4}{s^2+4^2} + \frac{s}{s^2+1^2}$$

$$4.) \mathcal{L}\{\sin t + e^{3t} \cos 4t\}$$

$$= \mathcal{L}\{\sin t\} + \mathcal{L}\{e^{3t} \cos 4t\}$$

$$= \frac{1}{s^2+1^2} + \frac{s-3}{(s-3)^2+4^2}$$

Inverse Laplace Transforms

Def : If $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s)$,
then $\mathcal{L}^{-1}\{F(s)\} = f(t)$ is the
inverse Laplace transform of $F(s)$.

Ex : 1.) $\mathcal{L}\{1\} = \frac{1}{s}$ so $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$

2.) $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2 + 3^2}$ so

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2 + 3^2}\right\} = \sin 3t$$

3.) $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$ so

$$\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3$$

4.) $\mathcal{L}\{e^{2t} \cos t\} = \frac{s-2}{(s-2)^2 + 1^2}$ so

$$\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2 + 1^2}\right\} = e^{2t} \cos t$$

Ex: Find the inverse Laplace transform for each $F(s)$.

$$1.) F(s) = \frac{1}{s-5} \rightarrow \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = e^{5t}$$

$$2.) F(s) = \frac{5!}{s^6} \rightarrow \mathcal{L}^{-1}\left\{\frac{5!}{s^6}\right\} = t^5$$

$$3.) F(s) = \frac{s}{s^2+25} \rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\} = \cos 5t$$

$$4.) F(s) = \frac{1}{s^2} + \frac{1}{s^2+16} \rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{1}{4} \cdot \frac{4}{s^2+4^2}\right\} = t + \frac{1}{4} \sin 4t$$

$$5.) F(s) = \frac{1}{(s+4)^2} \rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-(-4))^2}\right\} = te^{-4t}$$

$$6.) F(s) = \frac{1}{(s-1)^2+9} \rightarrow$$

$$\mathcal{L}^{-1}\left\{\frac{1}{3} \cdot \frac{3}{(s-1)^2+3^2}\right\} = \frac{1}{3} e^t \sin 3t$$

$$7.) F(s) = \frac{5}{s^2 - s - 6} = \frac{5}{(s-3)(s+2)}$$

$$= \frac{A}{s-3} + \frac{B}{s+2} \rightarrow$$

$$A(s+2) + B(s-3) = 5 :$$

$$\text{Let } s=3 : 5A = 5 \rightarrow A=1$$

$$\text{Let } s=-2 : -5B = 5 \rightarrow B = -1 ; \text{ then}$$

$$\mathcal{L}^{-1} \left\{ \frac{5}{s^2 - s - 6} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s-3} + \frac{-1}{s+2} \right\}$$

$$= e^{3t} - e^{-2t}$$

$$8.) F(s) = \frac{s+3}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1} \rightarrow$$

$$As(s^2+1) + B(s^2+1) + s^2(Cs+D) = s+3 :$$

$$\text{Let } s=0 : B=3$$

$$\text{Let } s=i : A(0) + B(0) + (-1)(Ci+D) = i+3 \rightarrow$$

$$(-C)i + (-D) = (1)i + (3) \rightarrow$$

$$-C=1 \rightarrow C=-1 \text{ and } -D=3 \rightarrow$$

$$D=-3$$

$$\text{Let } s=1 : 2'A + 3(2) + (-1+(-3)) = 4 \rightarrow$$

$$2A + 6 - 4 = 4 \rightarrow 2A = 2 \rightarrow A=1 ;$$

then

$$\mathcal{L}^{-1} \left\{ \frac{s+3}{s^2(s^2+1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{3}{s^2} + \frac{-s-3}{s^2+1} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{1}{s} + 3 \cdot \frac{1}{s^2} - \frac{s}{s^2+1} - 3 \cdot \frac{1}{s^2+1} \right\}$$

$$= 1 + 3t - \cos t - 3 \sin t$$

$$9.) F(s) = \frac{s}{s^2+2s+10} = \frac{s}{(s^2+2s+1)+9}$$

$$= \frac{s}{(s+1)^2+3^2} = \frac{s+1-1}{(s+1)^2+3^2}$$

$$= \frac{s+1}{(s+1)^2+3^2} - \frac{1}{3} \cdot \frac{3}{(s+1)^2+3^2} \rightarrow$$

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2+2s+10} \right\}$$

$$= \mathcal{L}^{-1} \left\{ \frac{s+1}{(s+1)^2+3^2} - \frac{1}{3} \cdot \frac{3}{(s+1)^2+3^2} \right\}$$

$$= e^{-t} \cdot \cos 3t - \frac{1}{3} \cdot e^{-t} \sin 3t$$