

Lecture 4

Math 17C

Laplace Transforms 4

HW #4

Laplace Transforms Applied to Systems of Differential Equations

Def: Let $x_1 = f(t)$ and $x_2 = g(t)$
be differentiable functions of t .
Then

$$\begin{cases} \frac{dx_1}{dt} = a \cdot x_1 + b \cdot x_2 \\ \frac{dx_2}{dt} = c \cdot x_1 + d \cdot x_2 \end{cases}$$

is called a 2x2 linear system of D.E.'s,
where a, b, c , and d are constants.

Note: Systems of D.E.'s can be
used to describe mixture
problems, spring problems,
predator/prey problems, etc.

Ex: Use Laplace transforms to solve
the following systems of D.E.'s.

$$1.) \begin{cases} \frac{dx_1}{dt} = x_2 & , & x_1(0) = 0 \\ \frac{dx_2}{dt} = -x_1 & , & x_2(0) = 1 \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\left\{\frac{dx_1}{dt}\right\} = \mathcal{L}\{x_2\} \\ \mathcal{L}\left\{\frac{dx_2}{dt}\right\} = -\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \begin{cases} s\mathcal{L}\{x_1\} - \cancel{x_1(0)}^0 = \mathcal{L}\{x_2\} \\ s\mathcal{L}\{x_2\} - \cancel{x_2(0)}^1 = -\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\{x_2\} = s\mathcal{L}\{x_1\} \rightarrow (s \cup B) \\ s\mathcal{L}\{x_2\} - 1 = -\mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow s(s\mathcal{L}\{x_1\}) - 1 = -\mathcal{L}\{x_1\}$$

$$\rightarrow (s^2 + 1)\mathcal{L}\{x_1\} = 1$$

$$\rightarrow \mathcal{L}\{x_1\} = \frac{1}{s^2 + 1} \rightarrow \boxed{x_1 = \sin t};$$

$$\text{and } \mathcal{L}\{x_2\} = s\mathcal{L}\{x_1\} = s \cdot \frac{1}{s^2 + 1} = \frac{s}{s^2 + 1}$$

$$\rightarrow \boxed{x_2 = \cos t}$$

$$2.) \begin{cases} \frac{dx_1}{dt} = x_1 + x_2 & , \quad x_1(0) = 3 \\ \frac{dx_2}{dt} = 2x_1 & , \quad x_2(0) = 0 \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\left\{\frac{dx_1}{dt}\right\} = \mathcal{L}\{x_1 + x_2\} \\ \mathcal{L}\left\{\frac{dx_2}{dt}\right\} = \mathcal{L}\{2x_1\} \end{cases}$$

$$\rightarrow \begin{cases} s \mathcal{L}\{x_1\} - \cancel{x_1(0)}^3 = \mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \\ s \mathcal{L}\{x_2\} - \cancel{x_2(0)}^0 = 2 \mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\{x_2\} = (s-1) \mathcal{L}\{x_1\} - 3 \rightarrow (S \cup B) \\ s \mathcal{L}\{x_2\} = 2 \mathcal{L}\{x_1\} \end{cases}$$

$$\rightarrow s((s-1) \mathcal{L}\{x_1\} - 3) = 2 \mathcal{L}\{x_1\}$$

$$\rightarrow s(s-1) \mathcal{L}\{x_1\} - 3s = 2 \mathcal{L}\{x_1\}$$

$$\rightarrow (s^2 - s - 2) \mathcal{L}\{x_1\} = 3s$$

$$\rightarrow \mathcal{L}\{x_1\} = \frac{3s}{s^2 - s - 2} = \frac{3s}{(s-2)(s+1)}$$

$$= \frac{A}{s-2} + \frac{B}{s+1} \rightarrow A(s+1) + B(s-2) = 3s:$$

$$\text{Let } s=2: 3A=6 \rightarrow A=2,$$

$$\text{Let } s=-1: -3B=-3 \rightarrow B=1; \text{ then}$$

$$\mathcal{L}\{x_1\} = \frac{2}{s-2} + \frac{1}{s+1} \rightarrow$$

$$\boxed{x_1 = 2e^{2t} + e^{-t}}; \text{ and}$$

$$s \mathcal{L}\{x_2\} = 2 \mathcal{L}\{x_1\} \rightarrow$$

$$s \mathcal{L}\{x_2\} = 2 \left(\frac{2}{s-2} + \frac{1}{s+1} \right) \rightarrow$$

$$\mathcal{L}\{x_2\} = \frac{4}{s(s-2)} + \frac{2}{s(s+1)}$$

$$\left(\frac{4}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \rightarrow A(s-2) + Bs = 4 : \right.$$

$$\text{Let } s=0: -2A=4 \rightarrow A=-2,$$

$$\text{Let } s=2: 2B=4 \rightarrow B=2; \text{ and}$$

$$\frac{2}{s(s+1)} = \frac{C}{s} + \frac{D}{s+1} \rightarrow C(s+1) + Ds = 2:$$

$$\text{Let } s=0: C=2,$$

$$\text{Let } s=-1: -D=2 \rightarrow D=-2) \rightarrow$$

$$\mathcal{L}\{x_2\} = \left(\frac{-2}{s} + \frac{2}{s-2} \right) + \left(\frac{2}{s} + \frac{-2}{s+1} \right) \rightarrow$$

$$\mathcal{L}\{x_2\} = \frac{2}{s-2} - \frac{2}{s+1} \rightarrow$$

$$x_2 = 2e^{2t} - 2e^{-t}$$

$$3.) \begin{cases} \frac{dx_1}{dt} = x_1 + x_2, & x_1(0) = 0 \\ \frac{dx_2}{dt} = -x_1 + x_2, & x_2(0) = 1 \end{cases}$$

$$\rightarrow \begin{cases} \mathcal{L}\left\{\frac{dx_1}{dt}\right\} = \mathcal{L}\{x_1 + x_2\} \\ \mathcal{L}\left\{\frac{dx_2}{dt}\right\} = \mathcal{L}\{-x_1 + x_2\} \end{cases}$$

$$\rightarrow \begin{cases} s\mathcal{L}\{x_1\} - \cancel{x_1(0)}^0 = \mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \\ s\mathcal{L}\{x_2\} - \cancel{x_2(0)}^1 = -\mathcal{L}\{x_1\} + \mathcal{L}\{x_2\} \end{cases}$$

$$\rightarrow \mathcal{L}\{x_2\} = (s-1)\mathcal{L}\{x_1\} \rightarrow (s \cup B)$$

$$\rightarrow s((s-1)\mathcal{L}\{x_1\}) - 1 = -\mathcal{L}\{x_1\} + (s-1)\mathcal{L}\{x_1\}$$

$$\rightarrow (s^2 - s)\mathcal{L}\{x_1\} + \mathcal{L}\{x_1\} - (s-1)\mathcal{L}\{x_1\} = 1$$

$$\rightarrow (s^2 - s + 1 - (s-1))\mathcal{L}\{x_1\} = 1$$

$$\rightarrow (s^2 - 2s + 2)\mathcal{L}\{x_1\} = 1$$

$$\rightarrow \mathcal{L}\{x_1\} = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1^2}$$

$$\rightarrow \boxed{x_1 = e^t \sin t} ; \text{ and}$$

$$\mathcal{L}\{x_2\} = (s-1)\mathcal{L}\{x_1\} = (s-1) \cdot \frac{1}{(s-1)^2 + 1} \rightarrow$$

$$\mathcal{L}\{x_2\} = \frac{s-1}{(s-1)^2 + 1^2} \rightarrow$$

$$\boxed{x_2 = e^t \cos t}$$