Math 17C
Koubа
Section 11.2

Mixture Problems and Systems of Differential Equations – an Example

SEE Handouts

$$\frac{dx_1}{dt} = (\text{Rate In}) - (\text{Rate Out}) \quad \left(\frac{\text{lbs.}}{\text{min.}}\right)$$

$$= \left(\frac{0 \text{ lbs.}}{\text{gal.}}\right)\left(\frac{20 \text{ gal.}}{\text{min.}}\right) + \left(\frac{x_2 \text{ lbs.}}{200 \text{ gal}}\right)\left(\frac{10 \text{ gal.}}{\text{min.}}\right) - \left(\frac{x_1 \text{ lbs.}}{100 \text{ gal}}\right)\left(\frac{30 \text{ gal.}}{\text{min.}}\right) \; ;$$

$$\frac{dx_2}{dt} = (\text{Rate In}) - (\text{Rate Out}) \quad \left(\frac{\text{lbs.}}{\text{min.}}\right)$$

$$= \left(\frac{x_1 \text{ lbs.}}{100 \text{ gal.}}\right)\left(\frac{30 \text{ gal.}}{\text{min.}}\right) - \left(\frac{x_2 \text{ lbs.}}{200 \text{ gal.}}\right)\left(\frac{10 \text{ gal.}}{\text{min.}}\right) - \left(\frac{x_2 \text{ lbs.}}{200 \text{ gal.}}\right)\left(\frac{20 \text{ gal.}}{\text{min.}}\right)$$

$$\rightarrow \begin{cases} \dfrac{dx_1}{dt} = -\dfrac{3}{10}x_1 + \dfrac{1}{20}x_2 \;, & x_1(0) = 40 \\[2ex] \dfrac{dx_2}{dt} = \dfrac{3}{10}x_1 - \dfrac{3}{20}x_2 \;, & x_2(0) = 100 \end{cases}$$

$$\rightarrow X' = \begin{bmatrix} -\frac{3}{10} & \frac{1}{20} \\ \frac{3}{10} & -\frac{3}{20} \end{bmatrix} X \rightarrow$$

$$\det(A - \lambda I) = \begin{bmatrix} -\frac{3}{10} - \lambda & \frac{1}{20} \\ \frac{3}{10} & -\frac{3}{20} - \lambda \end{bmatrix}$$

$$= \left(-\frac{3}{10} - \lambda\right)\left(-\frac{3}{20} - \lambda\right) - \left(\frac{3}{10}\right)\left(\frac{1}{20}\right) = 0 \quad \text{(mult. by 200)}$$

$$\rightarrow 10\left(-\frac{3}{10} - \lambda\right) \cdot 20\left(-\frac{3}{20} - \lambda\right) - 200\left(\frac{3}{200}\right) = (200)(0)$$

$$\rightarrow (-3 - 10\lambda)(-3 - 20\lambda) - 3 = 0$$

$$\rightarrow 200\lambda^2 + 90\lambda + 9 - 3 = 0$$

$$\rightarrow 200\lambda^2 + 90\lambda + 6 = 0$$

$$\rightarrow \boxed{100\lambda^2 + 45\lambda + 3 = 0} \quad ; \text{ then}$$

$$\lambda = \frac{-45 \pm \sqrt{(45)^2 - 4(100)(3)}}{2(100)} = \frac{-45 \pm \sqrt{825}}{200}$$

$\rightarrow \lambda = -0.08139$ or $\lambda = -0.36861$ (5 places.

Solve $(A - \lambda I)X = 0$:

<u>For $\lambda_1 = -0.08139$:</u>

$$\left[\begin{array}{cc|c} -0.21861 & 0.05 & 0 \\ 0.3 & -0.06861 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -0.22872 & 0 \\ 1 & -0.22870 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -0.22872 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow X_1 - 0.22872\, X_2 = 0$$

$\rightarrow$ let $X_2 = t$ any # $\rightarrow X_1 = 0.22872\, t$, so

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.22872t \\ t \end{bmatrix} = t\begin{bmatrix} 0.22872 \\ 1 \end{bmatrix}$$

so eigenvector is $V_1 = \begin{bmatrix} 0.22872 \\ 1 \end{bmatrix}$;

<u>For $\lambda_2 = -0.36861$:</u>

$$\left[\begin{array}{cc|c} 0.06861 & 0.05 & 0 \\ 0.3 & 0.21861 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0.72876 & 0 \\ 1 & 0.72870 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0.72876 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \quad x_1 + 0.72876\, x_2 = 0$$

$\to$ let $x_2 = t$ any # $\to$ $x_1 = -0.72876t$,

so $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -0.72876t \\ t \end{bmatrix} = t\begin{bmatrix} -0.72876 \\ 1 \end{bmatrix}$,

so eigenvector is $V_2 = \begin{bmatrix} -0.72876 \\ 1 \end{bmatrix}$;

then general solution

is $X = c_1 \begin{bmatrix} 0.22872 \\ 1 \end{bmatrix} e^{-0.08139t}$

$$+ c_2 \begin{bmatrix} -0.72876 \\ 1 \end{bmatrix} e^{-0.36861t}$$

OR

$$\begin{cases} x_1 = 0.22872c_1 e^{-0.08139t} - 0.72876\, c_2 e^{-0.36861t} \\ x_2 = c_1 e^{-0.08139t} + c_2 e^{-0.36861t} \end{cases};$$

then

$$\begin{cases} x_1(0) = 0.22872\,c_1 - 0.72876\,c_2 = 40 \\ x_2(0) = c_1 + c_2 = 100 \end{cases}$$

$\rightarrow$ $\underline{c_1 = 100 - c_2}$ $\rightarrow$ (SUB)

$0.22872\,(100 - c_2) - 0.72876\,c_2 = 40 \rightarrow$

$22.872 - 0.22872\,c_2 - 0.72876\,c_2 = 40 \rightarrow$

$-0.95748\,c_2 = 17.128 \longrightarrow$

$c_2 = -17.88862$ and

$c_1 = 117.88862$; then

$$\begin{cases} x_1 = 26.96349\,e^{-0.08139t} + 13.03651\,e^{-0.36861t} \\ x_2 = 117.88862\,e^{-0.08139t} - 17.88862\,e^{-0.36861t} \end{cases}$$

Questions:

1.) How much salt is in each tank when $t = 1$ min.? $t = 10$ min.?

$t = 1$ min. : $x_1 = 33.87316$ lbs.
$x_2 = 96.30028$ lbs.

$t = 10$ min : $x_1 = 12.27506$ lbs.
$x_2 = 51.79111$ lbs.

2.) How salt will be in each tank as $t \to \infty$?

$$\lim_{t\to\infty} x_1 = \lim_{t\to\infty} \left(26.96349e^{-0.08139t} + 13.03651e^{-0.36861t}\right)$$

$$= 26.96349\,(0) + 13.03651\,(0) = 0 \text{ lbs;}$$

$$\lim_{t\to\infty} x_2 = \lim_{t\to\infty} \left(117.88862e^{-0.08139t} - 17.88862e^{-0.36861t}\right)$$

$$= 117.88862\,(0) - 17.88862\,(0) = 0 \text{ lbs.}$$