Math 17C
Kouba
Worksheet 3.5

1.) Find the general solution to each of the following systems of differential equations. Write your answer in matrix (vector) form. (All eigenvalues are complex conjugates.)

a.) $X' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} X$

b.) $X' = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} X$

c.) $X' = \begin{pmatrix} -1 & 1 \\ -3 & 1 \end{pmatrix} X$

d.) $X' = \begin{pmatrix} 1 & 3 \\ -2 & -2 \end{pmatrix} X$

e.) $X' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X$

f.) $X' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X$

Worksheet 3.5

1.) a.) $X' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} X \rightarrow$

$\det(A-\lambda I) = \det \begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}$

$= (-1-\lambda)(-1-\lambda) - (1)(-1) = \lambda^2 + 2\lambda + 1 + 1 = \lambda^2 + 2\lambda + 2 = 0 \rightarrow$

$\lambda = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \dfrac{-2 \pm \sqrt{-4}}{2} = \dfrac{-2 \pm 2i}{2} = -1 \pm i$;

<u>For $\lambda = -1+i$</u> : $(A-\lambda I)X = 0 \rightarrow$

$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right] \rightarrow x_1 + i x_2 = 0$ so let $x_2 = t$ any # $\rightarrow$

$x_1 = -i x_2 = -it$ so

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$ and eigenvector is $\begin{bmatrix} -i \\ 1 \end{bmatrix}$;

solution is

$X = c \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-1+i)t} = c \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{-t} e^{it}$

$= c \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{-t} (\cos t + i \sin t)$

$= c \begin{bmatrix} -i \cos t + \sin t \\ \cos t + i \sin t \end{bmatrix} e^{-t}$

$= c \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{-t} + ci \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{-t}$

$= c_1 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{-t}$

b.) $X' = \begin{bmatrix} 2 & -1 \\ 3 & 0 \end{bmatrix} X \longrightarrow$

$\det(A-\lambda I) = \det \begin{bmatrix} 2-\lambda & -1 \\ 3 & -\lambda \end{bmatrix}$

$= (2-\lambda)(-\lambda) - (-1)(3) = \lambda^2 - 2\lambda + 3 = 0 \longrightarrow$

$$\lambda = \frac{2 \pm \sqrt{4-12}}{2} = \frac{2 \pm 2\sqrt{2}\,i}{2} = 1 \pm \sqrt{2}\,i \quad ;$$

For $\lambda = 1+\sqrt{2}\,i$: $(A-\lambda I)X = 0 \longrightarrow$

$$\left[\begin{array}{cc|c} 1-\sqrt{2}i & -1 & 0 \\ 3 & -1-\sqrt{2}i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1-\sqrt{2}i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$(1-\sqrt{2}i)x_1 - x_2 = 0$ so let $x_1 = t$ any # $\longrightarrow$

$x_2 = (1-\sqrt{2}i)x_1 = (1-\sqrt{2}i)t$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ (1-\sqrt{2}i)t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1-\sqrt{2}i \end{bmatrix}$ and eigenvector

is $\begin{bmatrix} 1 \\ 1-\sqrt{2}i \end{bmatrix}$; solution is

$$X = c\begin{bmatrix} 1 \\ 1-\sqrt{2}i \end{bmatrix} e^{(1+\sqrt{2}i)t} = c\begin{bmatrix} 1 \\ 1-\sqrt{2}i \end{bmatrix} e^{t} e^{\sqrt{2}ti}$$

$$= c\begin{bmatrix} 1 \\ 1-\sqrt{2}i \end{bmatrix} e^{t}(\cos\sqrt{2}t + i\sin\sqrt{2}t)$$

$$= c\begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ (1-\sqrt{2}i)\cos\sqrt{2}t + (i+\sqrt{2})\sin\sqrt{2}t \end{bmatrix} e^{t}$$

$$= c\begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} e^{t} + ci\begin{bmatrix} \sin\sqrt{2}t \\ -\sqrt{2}\cos\sqrt{2}t + \sin\sqrt{2}t \end{bmatrix} e^{t}$$

$$= c_1\begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t \end{bmatrix} e^{t} + c_2\begin{bmatrix} \sin\sqrt{2}t \\ \sin\sqrt{2}t - \sqrt{2}\cos\sqrt{2}t \end{bmatrix} e^{t}$$

c.) $X' = \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix} X \longrightarrow$

$\det(A-\lambda I) = \det\begin{bmatrix} -1-\lambda & 1 \\ -3 & 1-\lambda \end{bmatrix}$

$= (-1-\lambda)(1-\lambda) - (1)(-3) = \lambda^2 - 1 + 3 = \lambda^2 + 2 = 0 \longrightarrow$

$\lambda^2 = -2 \longrightarrow \lambda = \pm\sqrt{-2} = \pm\sqrt{2}\,i \quad ;$

For $\lambda = \sqrt{2}\, i$: $(A - \lambda I)X = 0 \longrightarrow$

$$\left[\begin{array}{cc|c} -1-\sqrt{2}i & 1 & 0 \\ -3 & 1-\sqrt{2}i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1-\sqrt{2}i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$(-1-\sqrt{2}i)x_1 + x_2 = 0$ so let $x_1 = t$ any # $\longrightarrow$
$x_2 = (1+\sqrt{2}i)x_1 = (1+\sqrt{2}i)t$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ (1+\sqrt{2}i)t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1+\sqrt{2}i \end{bmatrix}$ and eigenvector is $\begin{bmatrix} 1 \\ 1+\sqrt{2}i \end{bmatrix}$; solution is

$$X = c\begin{bmatrix} 1 \\ 1+\sqrt{2}i \end{bmatrix} e^{(\sqrt{2}i)t} = c\begin{bmatrix} 1 \\ 1+\sqrt{2}i \end{bmatrix} e^{(\sqrt{2}t)i}$$

$$= c\begin{bmatrix} 1 \\ 1+\sqrt{2}i \end{bmatrix} (\cos\sqrt{2}t + i\sin\sqrt{2}t)$$

$$= c\begin{bmatrix} \cos\sqrt{2}t + i\sin\sqrt{2}t \\ (1+\sqrt{2}i)\cos\sqrt{2}t + (i-\sqrt{2})\sin\sqrt{2}t \end{bmatrix}$$

$$= c\begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + ci\begin{bmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t + \sin\sqrt{2}t \end{bmatrix}$$

$$= c_1\begin{bmatrix} \cos\sqrt{2}t \\ \cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix} + c_2\begin{bmatrix} \sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t + \sin\sqrt{2}t \end{bmatrix}$$

d.) $X' = \begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix} X \longrightarrow$

$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 3 \\ -2 & -2-\lambda \end{bmatrix}$

$= (1-\lambda)(-2-\lambda) - (3)(-2) = \lambda^2 + \lambda - 2 + 6 = \lambda^2 + \lambda + 4 = 0 \rightarrow$

$\lambda = \dfrac{-1 \pm \sqrt{1-16}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{15}}{2} i$;

For $\lambda = \frac{-1}{2} + \frac{\sqrt{15}}{2} i$: $(A - \lambda I)X = 0 \longrightarrow$

$$\left[\begin{array}{cc|c} \frac{3}{2}-\frac{\sqrt{15}}{2}i & 3 & 0 \\ -2 & -\frac{3}{2}-\frac{\sqrt{15}}{2}i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & \frac{3}{4}+\frac{\sqrt{15}}{4}i & 0 \end{array}\right] \rightarrow$$

$x_1 + \left(\frac{3}{4}+\frac{\sqrt{15}}{4}i\right)x_2 = 0$ so let $x_2 = t$ any # $\rightarrow$

$x_1 = -\left(\frac{3}{4}+\frac{\sqrt{15}}{4}i\right)x_2 = \left(\frac{-3}{4}-\frac{\sqrt{15}}{4}i\right)t$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{-3}{4}-\frac{\sqrt{15}}{4}i\right)t \\ t \end{bmatrix} = -t\begin{bmatrix} \frac{3}{4}+\frac{\sqrt{15}}{4}i \\ -1 \end{bmatrix}$ and

eigenvector is $\begin{bmatrix} \frac{3}{4}+\frac{\sqrt{15}}{4}i \\ -1 \end{bmatrix}$; solution is

$$X = c\begin{bmatrix} \frac{3}{4}+\frac{\sqrt{15}}{4}i \\ -1 \end{bmatrix} e^{\left(\frac{-1}{2}+\frac{\sqrt{15}}{2}i\right)t}$$

$$= c\begin{bmatrix} 3/4+\frac{\sqrt{15}}{4}i \\ -1 \end{bmatrix} e^{\frac{-1}{2}t} e^{\left(\frac{\sqrt{15}}{2}t\right)i}$$

$$= c\begin{bmatrix} 3/4+\frac{\sqrt{15}}{4}i \\ -1 \end{bmatrix} e^{\frac{-1}{2}t}\left(\cos\frac{\sqrt{15}}{2}t + i\sin\frac{\sqrt{15}}{2}t\right)$$

$$= c\begin{bmatrix} \left(\frac{3}{4}+\frac{\sqrt{15}}{4}i\right)\cos\frac{\sqrt{15}}{2}t + \left(\frac{3}{4}i-\frac{\sqrt{15}}{4}\right)\sin\frac{\sqrt{15}}{2}t \\ -\cos\frac{\sqrt{15}}{2}t - i\sin\frac{\sqrt{15}}{2}t \end{bmatrix}$$

$$= c\begin{bmatrix} \frac{3}{4}\cos\frac{\sqrt{15}}{2}t - \frac{\sqrt{15}}{4}\sin\frac{\sqrt{15}}{2}t \\ -\cos\frac{\sqrt{15}}{2}t \end{bmatrix} e^{\frac{-1}{2}t}$$

$$+ ci\begin{bmatrix} \frac{\sqrt{15}}{4}\cos\frac{\sqrt{15}}{2}t + \frac{3}{4}\sin\frac{\sqrt{15}}{2}t \\ -\sin\frac{\sqrt{15}}{2}t \end{bmatrix} e^{\frac{-1}{2}t}$$

$$= c_1\begin{bmatrix} \frac{3}{4}\cos\frac{\sqrt{15}}{2}t - \frac{\sqrt{15}}{4}\sin\frac{\sqrt{15}}{2}t \\ -\cos\frac{\sqrt{15}}{2}t \end{bmatrix} e^{\frac{-1}{2}t}$$

$$+ c_2\begin{bmatrix} \frac{\sqrt{15}}{4}\cos\frac{\sqrt{15}}{2}t + \frac{3}{4}\sin\frac{\sqrt{15}}{2}t \\ -\sin\frac{\sqrt{15}}{2}t \end{bmatrix} e^{\frac{-1}{2}t}$$

e.) $X' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} X \longrightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)(1-\lambda) - (1)(-1) = \lambda^2 - 2\lambda + 1 + 1 = \lambda^2 - 2\lambda + 2 = 0 \rightarrow$

$\lambda = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$;

For $\lambda = 1+i$: $(A - \lambda I)X = 0 \longrightarrow$

$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right] \rightarrow x_1 + i x_2 = 0$ so let

$x_2 = t$ any # $\longrightarrow x_1 = -i x_2 = -it$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix} = -t \begin{bmatrix} i \\ -1 \end{bmatrix}$ and eigenvector is $\begin{bmatrix} i \\ -1 \end{bmatrix}$;

solution is

$X = c \begin{bmatrix} i \\ -1 \end{bmatrix} e^{(1+i)t} = c \begin{bmatrix} i \\ -1 \end{bmatrix} e^t e^{ti}$

$= c \begin{bmatrix} i \\ -1 \end{bmatrix} e^t (\cos t + i \sin t)$

$= c \begin{bmatrix} i \cos t - \sin t \\ -\cos t - i \sin t \end{bmatrix} e^t$

$= c \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} e^t + ci \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^t$

$= c_1 \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} e^t + c_2 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} e^t$

f.) $X' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X \longrightarrow$

$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$= \lambda^2 - (-1)(1) = \lambda^2 + 1 = 0 \longrightarrow \lambda^2 = -1 \longrightarrow$

$\lambda = \pm\sqrt{-1} = \pm i$;

<u>For $\lambda = i$</u> : $(A - \lambda I)X = \mathbf{0} \longrightarrow$

$$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & -i & 0 \end{array}\right] \longrightarrow x_1 - i x_2 = 0$$

so let $x_2 = t$ any # $\rightarrow x_1 = i x_2 = it$ and

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix} = t\begin{bmatrix} i \\ 1 \end{bmatrix}$ and eigenvector is $\begin{bmatrix} i \\ 1 \end{bmatrix}$;

solution is

$$X = c\begin{bmatrix} i \\ 1 \end{bmatrix} e^{it} = c\begin{bmatrix} i \\ 1 \end{bmatrix}(\cos t + i \sin t)$$

$$= c\begin{bmatrix} i\cos t - \sin t \\ \cos t + i \sin t \end{bmatrix}$$

$$= c\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + ci\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

$$= c_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$