Math 17C
Kouba
Exam 2 (Practice)

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO COPY ANSWERS FROM ANOTHER STUDENT'S EXAM. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO HAVE ANOTHER STUDENT TAKE YOUR EXAM FOR YOU. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive LITTLE or NO credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

5. Make sure that you have 9 pages, including the cover page.

6. You will be graded on proper use of limit, derivative, and integral notation.

7. You have until 8:50 a.m. sharp to finish the exam. Students who fail to stop working when time is called at the end of the exam MAY HAVE POINTS DEDUCTED from their score.

1.) (8 pts.) Find the Jacobi Matrix $Df(x, y)$ for $f(x, y) = \begin{pmatrix} \sin(x-y) \\ x^3y^2 \end{pmatrix}$.

$$Df(x,y) = \begin{bmatrix} \cos(x-y) & -\cos(x-y) \\ 3x^2y^2 & 2x^3y \end{bmatrix}$$

2.) (5 pts.) The point $(0, 0)$ is an equilibrium for the following system of differential equations. Determine if $(0, 0)$ is an unstable or stable equilibrium. Then categorize $(0, 0)$ as a sink, source, saddle, stable spiral, unstable spiral, or neutral spiral.

$$X' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$$

$$\det(A-\lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

$$= \lambda^2 - (1)(-1) = \lambda^2 + 1 = 0 \longrightarrow$$

$$\lambda^2 = -1 \longrightarrow \lambda = \pm i = 0 \pm i$$

so $a = 0$ and $(0,0)$ is neutral spiral (unstable)

3.) (12 pts.) Solve the following system of differential equations with initial conditions. Write your answer in matrix (vector) form.

$$X' = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} X \quad \text{and} \quad x_1(0) = 0,\ x_2(0) = 1$$

$\det(A - \lambda I) = 0 \rightarrow \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = 0 \rightarrow$

$(2-\lambda)^2 - (-1)(-1) = \lambda^2 - 4\lambda + 4 - 1 = \lambda^2 - 4\lambda + 3 = 0 \rightarrow$

$(\lambda - 3)(\lambda - 1) = 0 \rightarrow \lambda = 1, \lambda = 3$;

For $\lambda = 1$: Solve $(A - \lambda I)X = 0$ for $X \rightarrow$

$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 - x_2 = 0$ so

let $x_2 = t$ (any #) $\rightarrow x_1 = x_2 = t$ so

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so eigenvector is

$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

For $\lambda = 3$: Solve $(A - \lambda I)X = 0$ for $X \rightarrow$

$\left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x_1 + x_2 = 0$ so let $x_2 = t$ (any #) $\rightarrow$

$x_1 = -x_2 = -t$ so $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so eigenvector is $V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Then

sol. is $X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$ and

$X(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 1 \end{cases} \rightarrow 2c_1 = 1$

$\rightarrow c_1 = \frac{1}{2} \rightarrow c_2 = \frac{1}{2} \rightarrow$

$$X = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$$

4.) The position (x_1, x_2) of a particle at time t is given parametrically by

$$\begin{cases} x_1 = t^2 + 2t \\ x_2 = 2t - t^2 \qquad \text{for } 0 \le t \le 2 \end{cases}$$

and its graph is given below. For $t = 1$ determine

a.) and plot the point (x_1, x_2).

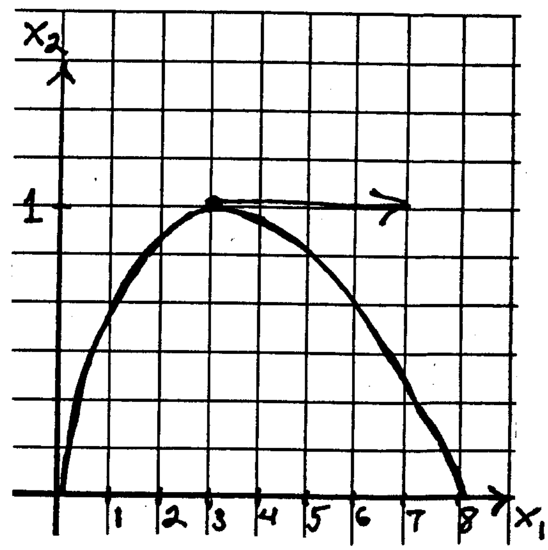
$(x_1, x_2) = (3, 1)$

b.) the slope of the graph. $\text{SLOPE} = \frac{dx_2}{dx_1} = \frac{x_2'}{x_1'} = \frac{2-2t}{2t+2}$, for $t=1$

$m = \frac{x_2'}{x_1'} = \frac{0}{4} = 0$

c.) and sketch a direction vector. Velocity vector pts. in direction of motion:

$X' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$



d.) the speed of the particle.

$\frac{ds}{dt} = \sqrt{(x_1')^2 + (x_2')^2}$

$= \sqrt{(4)^2 + (0)^2}$

$= \sqrt{16}$

$= 4$

5.) (8 pts.) The position (x_1, x_2) of a particle at time t is given parametrically by the following. Eliminate t and write the path as an equation in only x_1 and x_2 . Then sketch the graph of the path in the x_1x_2-plane, indicating the direction of motion of the particle.

$$\begin{cases} x_1 = t^2 \\ x_2 = t^6 + 1, \qquad \text{for } -1 \le t \le 2 \end{cases}$$

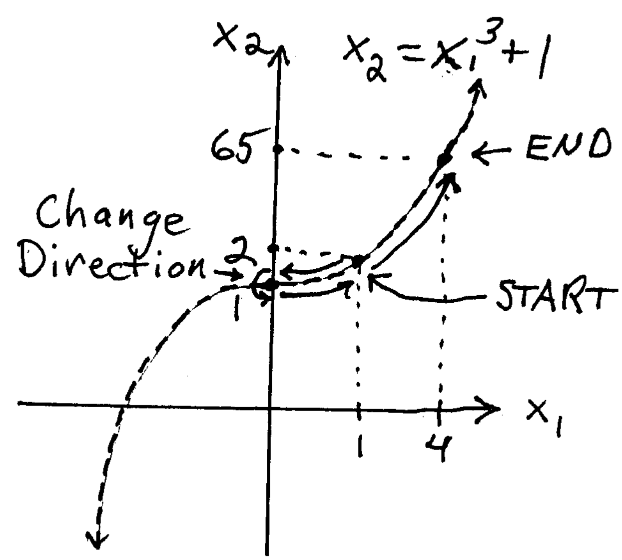
$x_2 = t^6 + 1 = (t^2)^3 + 1 = x_1^3 + 1$

$\rightarrow x_2 = x_1^3 + 1$

$t = -1: \quad x_1 = 1, \; x_2 = 2 \quad$ (START)

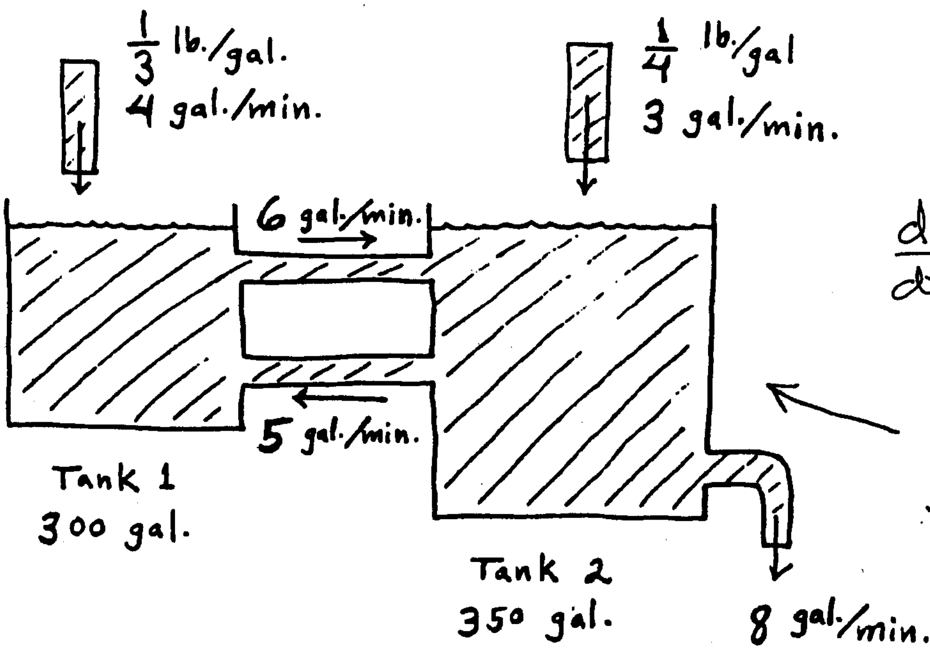
$t = 0: \quad x_1 = 0, \; x_2 = 1 \quad$ (change Direction)

$t = 2: \quad x_1 = 4, \; x_2 = 65 \quad$ (END)



6.) (10 pts.) Consider the two tanks containing water and salt mixtures and connected as shown in the diagram. Tank 1 holds 300 gallons of mixture. Tank 2 holds 350 gallons of mixture. Let x_1 and x_2 represent the pounds of salt in Tank 1 and Tank 2, resp., at time t. Initially, Tank 1 contains 35 pounds of salt and Tank 2 contains 40 pounds of salt. The mixture in each tank is kept uniform by stirring, and the mixtures are pumped from each tank to the other at the rates indicated in the figure. In addition, a mixture containing 1/3 pound of salt per gallon is pumped into Tank 1 at 4 gal./min.; a mixture containing 1/4 pound of salt per gallon is pumped into Tank 2 at 3 gal./min.; the mixture leaves Tank 2 at 8 gal./min. SET UP, BUT DO NOT SOLVE, a sytem of differential equations with initial conditions, which represents the amount of salt in each tank. PAY CLOSE ATTENTION TO FLOW RATES IN AND OUT OF EACH TANK !!!!

gains 3 gal./min.



$\frac{dx}{dt}$ = (rate in) − (rate out)

loses 4 gal./min.

$$\frac{dx_1}{dt} = \left[(4)\cdot\left(\frac{1}{3}\right) + (5)\cdot\frac{x_2}{350-4t}\right] - \left[(6)\cdot\frac{x_1}{300+3t}\right]$$

and $x_1(0) = 35$ lbs.;

$$\frac{dx_2}{dt} = \left[(3)\left(\frac{1}{4}\right) + (6)\left(\frac{x_1}{300+3t}\right)\right] - \left[(5)\cdot\left(\frac{x_2}{350-4t}\right) + (8)\cdot\left(\frac{x_2}{350-4t}\right)\right]$$

and $x_2(0) = 40$ lbs.

7.) (10 pts.) (New Drug Dissipation Equations) Caffeine, a bitter white crystalline xanthine alkoloid that acts as a psychoactive stimulant drug, is consumed daily by 90% of Americans. Assume that 100 mg. of caffeine is ingested by a coffee drinker. Let x_1 and x_2 be the mg. of caffeine in the person's body tissue and urinary tract, resp., at time t hours. Assume that the dissipation of caffeine satisfies the following (new) nonlinear system of differential equations given below. Solve this system for x_1 and x_2 :

$$\begin{cases} \dfrac{dx_1}{dt} = -(1/500)x_1^2 \quad \text{and} \quad x_1(0) = 100 \text{ mg.} \\ \dfrac{dx_2}{dt} = (1/500)x_1^2 \quad \text{and} \quad x_2(0) = 0 \text{ mg.} \qquad \text{and assume that} \quad x_1 + x_2 = 100\ . \end{cases}$$

$$\frac{dx_1}{dt} = \frac{-1}{500} x_1^2 \rightarrow \int \frac{-1}{x_1^2}\, dx_1 = \int \frac{1}{500}\, dt \rightarrow$$

$$\frac{1}{x_1} = \frac{1}{500} t + c \quad \text{and } t=0,\ x_1 = 100 \rightarrow$$

$$\frac{1}{100} = 0 + c \rightarrow c = \frac{1}{100} \rightarrow \frac{1}{x_1} = \frac{1}{500} t + \frac{1}{100} = \frac{t+5}{500} \rightarrow$$

$$\boxed{x_1 = \frac{500}{t+5}} \quad \text{and} \quad x_2 = 100 - x_1 \rightarrow$$

$$\boxed{x_2 = 100 - \frac{500}{t+5}}$$

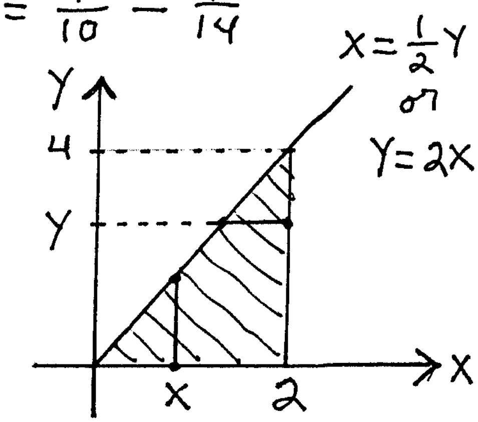
8.) (6 pts. each) Evaluate each double integral.

a.) $\displaystyle\int_0^1 \int_{x^2}^{x} x^2 y\, dy\, dx = \int_0^1 \left(x^2 \cdot \frac{1}{2} y^2 \Big|_{y=x^2}^{y=x} \right) dx$

$$= \int_0^1 \left(\frac{1}{2}x^4 - \frac{1}{2}x^6 \right) dx = \left(\frac{1}{10}x^5 - \frac{1}{14}x^7 \right)\Big|_0^1 = \frac{1}{10} - \frac{1}{14}$$

$$= \frac{14-10}{140} = \frac{4}{140} = \frac{1}{35}$$

b.) $\displaystyle\int_0^4 \int_{(1/2)y}^{2} \cos(x^2)\, dx\, dy$ (SWITCH ORDER)



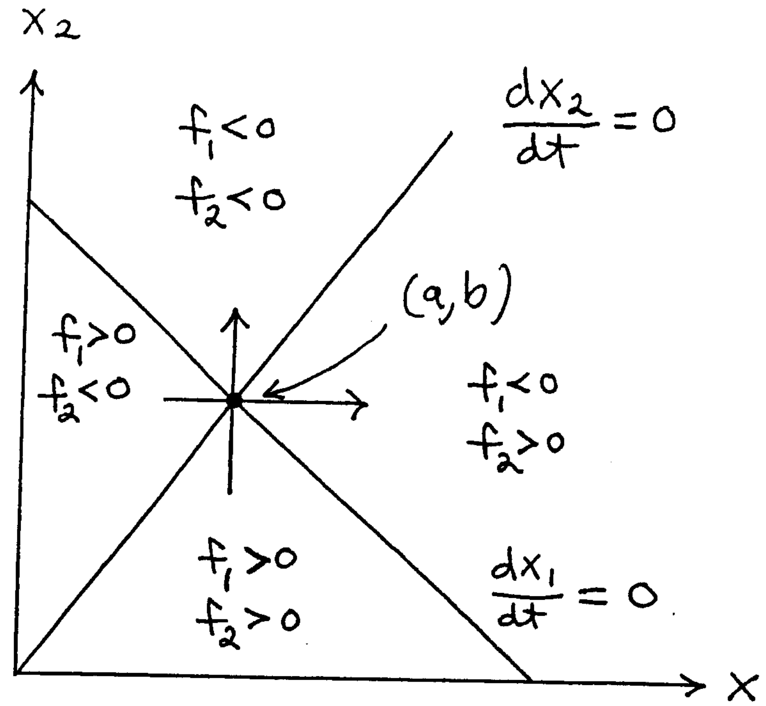
$$= \int_0^2 \int_0^{2x} \cos(x^2)\, dy\, dx$$

$$= \int_0^2 \left(y \cos(x^2) \Big|_{y=0}^{y=2x} \right) dx$$

$$= \int_0^2 2x \cos(x^2)\, dx = \sin(x^2) \Big|_0^2$$

$$= \sin 4 - \overset{0}{\cancel{\sin 0}} = \sin 4$$

9.) (10 pts.) (Graphical Method) Consider the given graph of zero isoclines with equilibrium $(a.b)$ and with a sign chart for f_1 and f_2. Use this sign chart to create a signed Jacobi Matrix. Then determine if the equilibrium is stable or unstable.



<u>along x_1 arrow</u>:

f_1 goes from $(+)$ to $(-)$ so f_1 is ↓ and $\frac{\partial f_1}{\partial x_1}$ is $\boxed{(-)}$;

f_2 goes from $(-)$ to $(+)$ so f_2 is ↑ and $\frac{\partial f_2}{\partial x_1}$ is $\boxed{(+)}$;

<u>along x_2 arrow</u>:

f_1 goes from $(+)$ to $(-)$, so f_1 is ↓ and $\frac{\partial f_1}{\partial x_2}$ is $\boxed{(-)}$;

f_2 goes from $(+)$ to $(-)$, so f_2 is ↓ and $\frac{\partial f_2}{\partial x_2}$ is $\boxed{(-)}$;

$$Df(a,b) = \begin{bmatrix} - & - \\ + & - \end{bmatrix} = A; \quad trA = (-)+(-) = (-);$$

$\det A = (-)(-) - (+)(-) = (+) - (-) = (+)$, so real parts of both eigenvalues are negative so (a,b) is <u>stable</u> equilibrium.

10.) Consider the following nonlinear system of differential equations.

$$\frac{dx_1}{dt} = x_1 - x_2$$
$$\frac{dx_2}{dt} = x_1 x_2 - x_2$$

a.) (3 pts.) Determine all zero isoclines for this system.

$x_1 - x_2 = 0 \rightarrow \boxed{x_1 = x_2}$; $(x_1 - 1)x_2 = 0 \rightarrow$ $\boxed{x_2 = 0}$ or $\boxed{x_1 = 1}$; zero isoclines are boxed.

b.) (3 pts.) Determine all equilibria for this system.

$(0,0)$ and $(1,1)$

c.) (10 pts.) For each equilibrium use the analytical approach (Jacobi Matrix and eigenvalues) to determine stability and classify it, or state that this method is inconclusive.

$$Df(x_1, x_2) = \begin{bmatrix} 1 & -1 \\ x_2 & x_1 - 1 \end{bmatrix};$$

a.) For $(0,0)$: $Df(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = A$ then

$\det(A - \lambda I) = 0 \rightarrow \det\begin{bmatrix} 1-\lambda & -1 \\ 0 & -1-\lambda \end{bmatrix} = (1-\lambda)(-1-\lambda) = 0$

$\rightarrow \lambda_1 = 1,\ \lambda_2 = -1$ so $(0,0)$ is <u>unstable</u>

(SADDLE)

b.) For $(1,1)$: $Df(1,1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = A$ then

$\det(A - \lambda I) = 0 \rightarrow \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda + 1 = 0 \rightarrow$

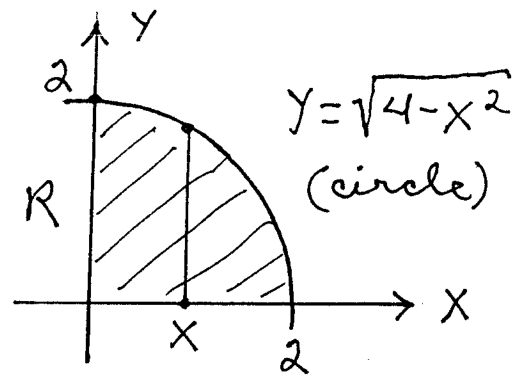
$$\lambda = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$a = \frac{1}{2}$ is $(+)$

so $(1,1)$ is <u>unstable</u> (UNSTABLE SPIRAL)

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Compute the average value of $f(x,y) = 5 - x$ defined on the region in the first quadrant bounded by the graphs of $x = 0$, $y = 0$, and $y = \sqrt{4 - x^2}$.



$$\text{Area } R = \int_0^2 \sqrt{4-X^2}\, dX$$

$$= \frac{1}{4}\pi (2)^2 = \pi \quad ;$$

$$AVE = \frac{1}{\text{Area } R} \int_0^2 \int_0^{\sqrt{4-X^2}} (5-X)\, dY\, dX$$

$$= \frac{1}{\pi} \int_0^2 (5Y - XY) \Big|_{Y=0}^{Y=\sqrt{4-X^2}} dX$$

$$= \frac{1}{\pi} \int_0^2 \left(5\sqrt{4-X^2} - X\sqrt{4-X^2}\right) dX$$

$$= \frac{1}{\pi} \left\{ 5 \int_0^2 \sqrt{4-X^2}\, dX - \int_0^2 X\sqrt{4-X^2}\, dX \right\}$$

$$= \frac{1}{\pi} \left\{ 5 \cdot \pi - \frac{-1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} (4-X^2)^{3/2} \Big|_0^2 \right\}$$

$$= \frac{1}{\pi} \left\{ 5\pi + \frac{1}{3}\left(0 - 4^{3/2}\right) \right\}$$

$$= \frac{1}{\pi} \left\{ 5\pi - \frac{8}{3} \right\}$$