Math 21A
Kouba
Exam 3

Please PRINT your name here : KEY

Your Exam ID Number ____________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

2. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.

3. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

4. Make sure that you have 6 pages, including the cover page.

5. On optimization (maximum/minimum) problems use a sign chart to verify the minimum or maximum and list optimal values for ALL variables used in the problem.

6. You will be graded on proper use of limit and derivative notation.

7. Put units on answers where units are appropriate.

8. You have until 9:50 a.m. sharp to finish the exam.

1.) (7 pts. each) Determine $y' = \dfrac{dy}{dx}$. DO NOT SIMPLIFY ANSWERS.

a.) $y = x^2 \ln(3x + 4)$

$$y' = x^2 \cdot \frac{1}{3x+4} \cdot (3) + 2x \cdot \ln(3x+4)$$

b.) $y = x^{\sec x}$ $\xrightarrow{\ln}$ $\ln y = \ln x^{\sec x} = \sec x \cdot \ln x$

$$\xrightarrow{D} \frac{1}{y} y' = \sec x \cdot \frac{1}{x} + \sec x \cdot \tan x \cdot \ln x \longrightarrow$$

$$y' = x^{\sec x} \cdot \left(\frac{\sec x}{x} + \sec x \cdot \tan x \cdot \ln x \right)$$

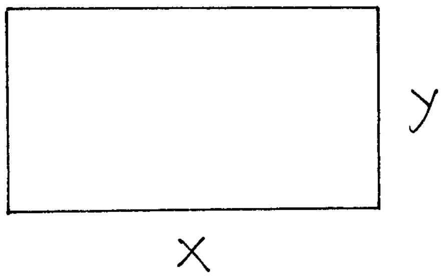
2.) (10 pts.) An initial deposit of \$1000 in a bank account grows to \$3,000 in 7 years. If the interest is compounded continuously, what is the annual interest rate r ?

$$A = Pe^{rt} \longrightarrow 3000 = 1000\, e^{r(7)} \longrightarrow$$

$$3 = e^{7r} \longrightarrow \ln 3 = \ln e^{7r} = 7r \longrightarrow$$

$$r = \frac{1}{7} \ln 3 \approx 15.69\ \%$$

3.) (11 pts.) Determine the dimensions of the rectangle of largest area with a fixed perimeter of 20 feet.



$2X + 2Y = 20 \rightarrow$

$X + Y = 10 \rightarrow$

$y = 10 - X$;

maximize area $A = XY = x(10-x) = 10X - x^2$

$\rightarrow A = 10X - X^2$

$\xrightarrow{D} A' = 10 - 2X = 0$

$\rightarrow X = 5$

$+ \quad 0 \quad -$ A'

$X = 5$ ft.

$y = 5$ ft.

max. $A = 25$ ft.^{2}

4.) (11 pts.) Find an equation of the line which is perpendicular to the graph of $y^2 + xy = 9$ at the point $x = 0, y = 3$.

$y^2 + xy = 9 \xrightarrow{D} 2YY' + XY' + 1 \cdot Y = 0 \rightarrow$

$(2y + x)y' = -y \rightarrow y' = \dfrac{-y}{2Y+X}$ at $X=0, Y=3$

$\rightarrow y' = \dfrac{-3}{6} = \dfrac{-1}{2}$ so $\perp$ slope $m = 2$

$\rightarrow \boxed{y = 2X + 3}$

5.) (10 pts.) Use Newton's Method to estimate the solution to $x^3 = x + 5$ to two decimal places.

$x^3 - x - 5 = 0 \rightarrow f(x) = x^3 - x - 5 \xrightarrow{D}$

$f'(x) = 3x^2 - 1$ ~~so~~ Newton:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 5}{3x_n^2 - 1}$$

$$= \frac{3x_n^3 - \cancel{x_n} - x_n^3 + \cancel{x_n} + 5}{3x_n^2 - 1} = \frac{2x_n^3 + 5}{3x_n^2 - 1} \rightarrow$$

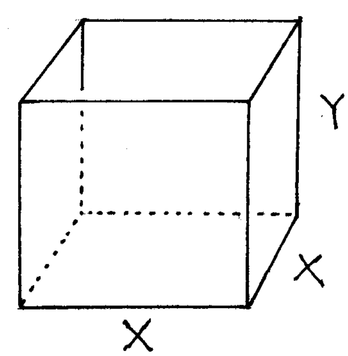
$$x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 1}$$; let $x_1 = 2$ then

$$x_2 = \frac{2x_1^3 + 5}{3x_1^2 - 1} \approx 1.909 \rightarrow x_3 = \frac{2x_2^3 + 5}{3x_2^2 - 1} \approx 1.904$$ so

solution $r \approx 1.90$

6.) (11 pts.) You are to construct an open (no top) rectangular box with a square base. Material for the base costs 30 cents/in.^{2} and material for the sides costs 10 cents/in.^{2}. If the box's volume is required to be 12 in.^{3}, what dimensions will result in the least expensive box ?

Volume $x^2 y = 12 \rightarrow y = \frac{12}{x^2}$;



minimize cost

$$C = C_{bott.} + C_{sides}$$

$$= 30(x^2) + 10(4xy)$$

$$= 30x^2 + 40x\left(\frac{12}{x^2}\right) = 30x^2 + \frac{480}{x} \xrightarrow{D}$$

$$C' = 60x - \frac{480}{x^2} = \frac{60x^3 - 480}{x^2} = \frac{60(x^3 - 8)}{x^2} = 0$$

$-$ 0 $+$ C'

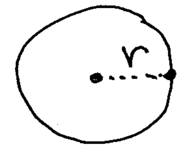
$x = 2$ in.

$y = 3$ in.

min. $C = 360¢ = \$3.60$

7.) (11 pts.) The volume of a sphere is changing at the rate of 64π ft.^{3}/min. At what rate is the sphere's surface area changing when the sphere's radius is $r = 2$ feet ? RECALL: A sphere of radius r has volume $V = (4/3)\pi r^3$ and surface area $S = 4\pi r^2$.

$\frac{dV}{dt} = 64\pi \frac{ft.^3}{min.}$, find $\frac{dS}{dt}$ when



$r = 2$ ft. ; $V = \frac{4}{3}\pi r^3 \xrightarrow{D} \frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 \frac{dr}{dt} \rightarrow$

$64\pi = 4\pi(2)^2 \cdot \frac{dr}{dt} \rightarrow \frac{dr}{dt} = \frac{64\cancel{\pi}}{16\cancel{\pi}} = 4$ ft./min. ;

$S = 4\pi r^2 \rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt} = 8\pi(2)(4) \rightarrow$

$\frac{dS}{dt} = 64\pi$ ft.^{2}/min.

8.) (10 pts.) Use a differential to estimate the value of $(84)^{1/4}$.

Let $f(x) = x^{1/4}$, $x: 81 \rightarrow 84$ and $\Delta x = 3$,

$f'(x) = \frac{1}{4} x^{-3/4}$; then

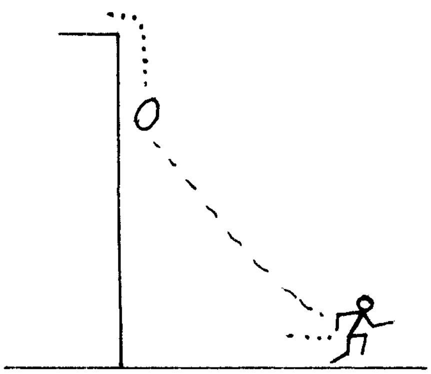
$\Delta f = f(84) - f(81) = (84)^{1/4} - (81)^{1/4} = (84)^{1/4} - 3$;

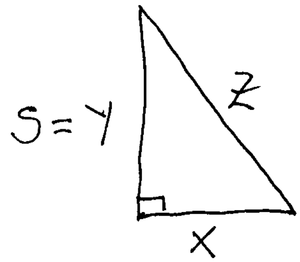
$df = f'(81) \cdot \Delta x = \frac{1}{4} \cdot \frac{1}{(81)^{3/4}} \cdot (3) = \frac{1}{4}\left(\frac{1}{27}\right)(3) = \frac{1}{36}$;

and $\Delta f \approx df$ so

$(84)^{1/4} - 3 \approx \frac{1}{36} \rightarrow (84)^{1/4} \approx 3\frac{1}{36} \approx 3.0277$

9.) (12 pts.) You are standing at the base of a 200 foot high building. An ostrich egg is dropped from the top of the building directly above you and you sprint away at 14 ft./sec. At what rate is the distance between you and the egg changing 3 seconds later ?





Height of egg is :

$s'' = -32 \rightarrow$

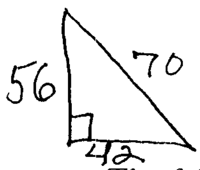
$s' = -32t + C$ $(s'(0) = 0 \rightarrow C = 0) \rightarrow \boxed{s' = -32t}$;

$s = -16t^2 + C$ $(s(0) = 200 \rightarrow C = 200)$

$\rightarrow \boxed{s = 200 - 16t^2}$; given

$\frac{dx}{dt} = 14$ ft./sec., find $\frac{dz}{dt}$ when $t = 3$ sec. ;

$x^2 + s^2 = z^2 \xrightarrow{D} 2x \cdot \frac{dx}{dt} + 2s \cdot \frac{ds}{dt} = 2z \cdot \frac{dz}{dt} \rightarrow$

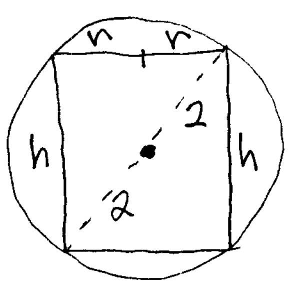


$(42)(14) + (56)(-96) = (70)\frac{dz}{dt} \rightarrow$

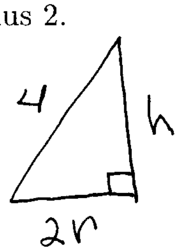
$\frac{dz}{dt} = -68.4$ ft./sec.

The following EXTRA CREDIT PROBLEM is worth 12 points. It is OPTIONAL.

1.) Determine the radius and height of the cylinder of maximum volume which can be inscribed in a sphere of radius 2.



SIDE VIEW



$(2r)^2 + h^2 = 4^2 \rightarrow$

$4r^2 + h^2 = 16 \rightarrow$

$4r^2 = 16 - h^2 \rightarrow$

$r^2 = \frac{1}{4}(16 - h^2)$;

maximize volume

$V = \pi r^2 h = \pi \cdot \frac{1}{4}(16 - h^2)h = \frac{\pi}{4}(16h - h^3) \rightarrow$

$V' = \frac{\pi}{4}(16 - 3h^2) = 0 \rightarrow h = \frac{4}{\sqrt{3}}$ and

$r = \frac{1}{2}\left(16 - \frac{16}{3}\right)^{1/2} = \sqrt{8/3}$

$+ \quad 0 \quad -$ $\quad V'$

$h = 4/\sqrt{3}$

$r = \sqrt{8/3}$ and max. $V = \frac{32}{3\sqrt{3}}\pi$