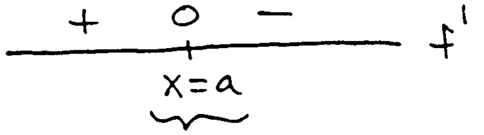
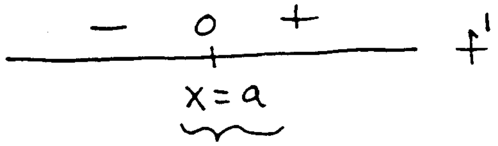
Graphing Using First and Second Derivatives

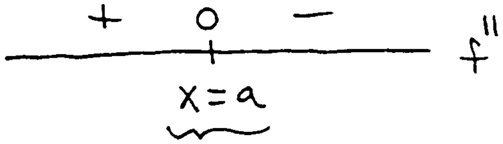
1. If f' is $+$, then f is increasing $(\uparrow)$.
2. If f' is $-$, then f is decreasing $(\downarrow)$.
3. If f'' is $+$ (means f' is $\uparrow$), then f is concave up $(\cup)$.
4. If f'' is $-$ (means f' is $\downarrow$), then f is concave down $(\cap)$.



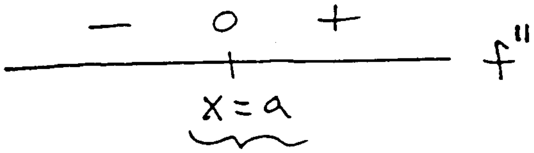
relative (or absolute) maximum



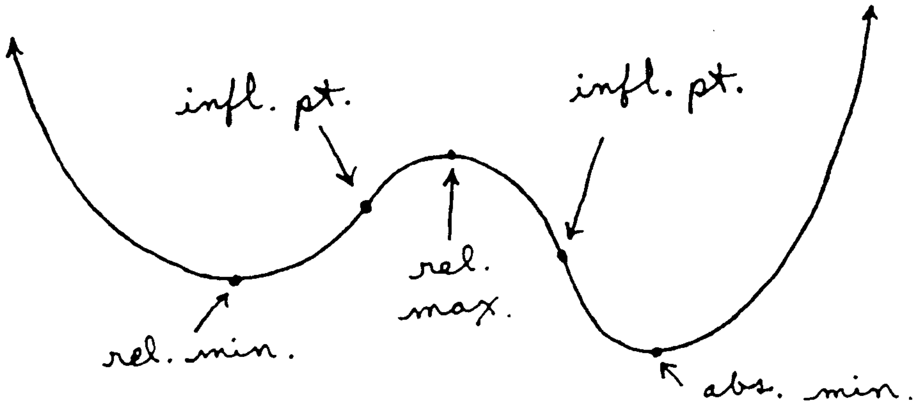
relative (or absolute) minimum



inflection point



inflection point



Math 21A
Kouba
Graphing

For each of the following functions begin by finding the domain of the function. Determine all relative and absolute maximum and minimum values and inflection points. State clearly the intervals on which the function is increasing (↑), decreasing (↓), concave up (∪), and concave down (∩). Determine all vertical and horizontal asymptotes (when appropriate) and x- and y-intercepts. Neatly sketch the graph.

Example 1: $f(x) = (x-1)^3(x-5)$ domain: all x-values

$$f'(x) = (x-1)^3 \cdot (1) + 3(x-1)^2 \cdot (x-5)$$
$$= (x-1)^2[(x-1) + 3(x-5)]$$
$$= (x-1)^2[4x-16] = 0$$

$-$ 0 $-$ 0 $+$ Y'

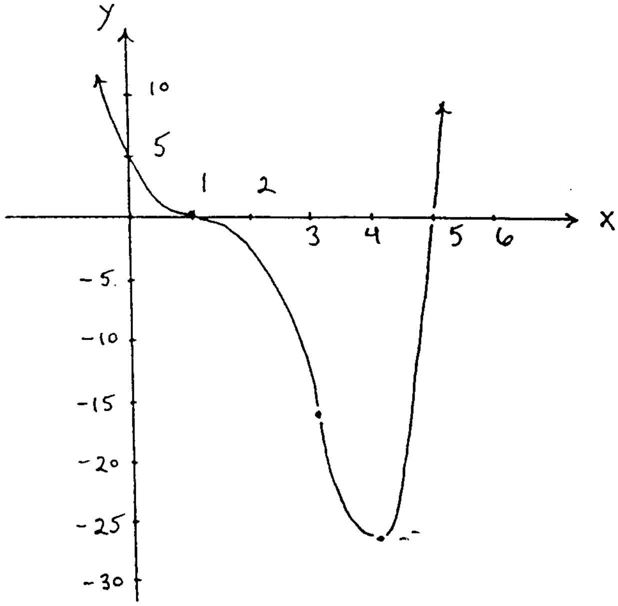
? $\left\{ \begin{matrix} X=1 \\ Y=0 \end{matrix} \right.$ $\left. \begin{matrix} X=4 \\ Y=-27 \end{matrix} \right\}$ abs. min.

$$f''(x) = (x-1)^2(4) + 2(x-1)[4x-16]$$
$$= 4(x-1)[(x-1) + 2(x-4)]$$
$$= 4(x-1)[3x-9] = 0$$

$+$ 0 $-$ 0 $+$ Y''

inf. pt. $\left\{ \begin{matrix} X=1 \\ Y=0 \end{matrix} \right.$ $\left. \begin{matrix} X=3 \\ Y=-16 \end{matrix} \right\}$ inf. pt.

f is ↑ for $x > 4$,
f is ↓ for $x < 4$,
f is ∪ for $x < 1$, $x > 3$,
f is ∩ for $1 < x 3$,
$Y = 0$: $X = 1$, $X = 5$
$X = 0$: $Y = 5$



Example 2: $Y = 3X^{2/3} - 2X$ domain: all x-values

$$Y' = 3 \cdot \frac{2}{3} X^{-1/3} - 2 = 2X^{-1/3} - 2$$
$$= 2\left(\frac{1}{X^{1/3}} - 1\right) = 2\left(\frac{1 - X^{1/3}}{X^{1/3}}\right) = 0$$

$-$ ¦ $+$ 0 $-$ Y'

rel. min. $\left\{ \begin{matrix} X=0 \\ Y=0 \end{matrix} \right.$ $\left. \begin{matrix} X=1 \\ Y=1 \end{matrix} \right\}$ rel. max.

$$Y'' = 2 \cdot \frac{-1}{3} X^{-4/3} = \frac{-2}{3X^{4/3}}$$

$- -$ ¦ $- -$ Y''

$X = 0$
$Y = 0$

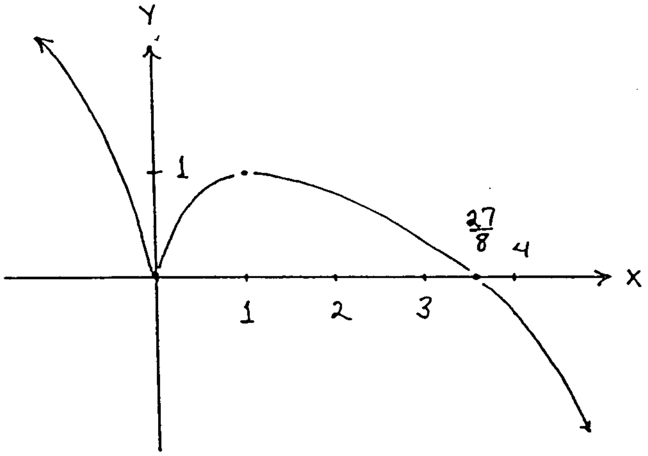
Y is ↑ for $0<x<1$,
Y is ↓ for $x<0,\ x>1$,
Y is $\cap$ for $x<0,\ x>0$

$x=0 : Y=0$

$Y=0 : \quad 3x^{2/3}-2x=0$

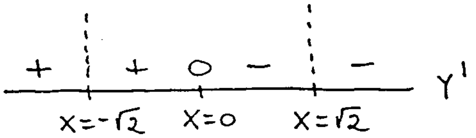
$$\to x^{2/3}\left(3-2x^{1/3}\right)=0$$

$$\begin{array}{cc} \downarrow & \downarrow \\ x=0 & x=\frac{27}{8} \end{array}$$



Example 3: $Y=\dfrac{x^2+1}{x^2-2}$ domain : all $x \neq \pm\sqrt{2}$

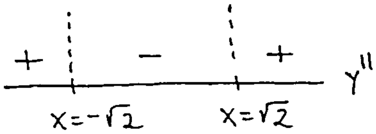
$$Y' = \frac{(x^2-2)(2x)-(x^2+1)(2x)}{(x^2-2)^2} = \frac{-6x}{(x^2-2)^2}=0$$

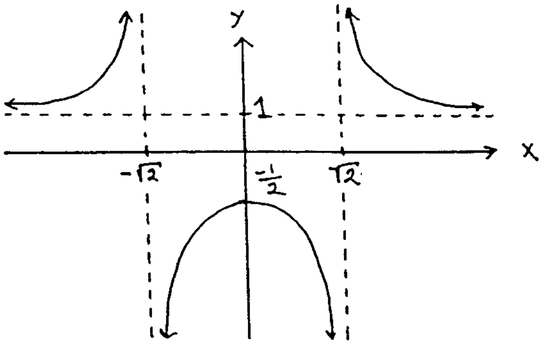


$Y=\frac{-1}{2}$
rel.
max.

$$Y'' = \frac{(x^2-2)^2(-6)-(-6x)\cdot 2(x^2-2)\cdot 2x}{(x^2-2)^4}$$

$$= \frac{-6(x^2-2)\cdot\left[(x^2-2)-4x^2\right]}{(x^2-2)^4} = \frac{6(2+3x^2)}{(x^2-2)^3}=0$$





Y is ↑ for $x<-\sqrt{2},\ -\sqrt{2}<x<0$,
Y is ↓ for $0<x<\sqrt{2},\ x>\sqrt{2}$,
Y is $\cup$ for $x<-\sqrt{2},\ x>\sqrt{2}$,
Y is $\cap$ for $-\sqrt{2}<x<\sqrt{2}$

$x=0 : \ Y=-1/2$

$Y=0$: none

$$\left.\begin{array}{l} \lim\limits_{x\to+\sqrt{2}^+} \dfrac{x^2+1}{x^2-2} = +\infty, \quad \lim\limits_{x\to+\sqrt{2}^-} \dfrac{x^2+1}{x^2-2} = -\infty \\ \lim\limits_{x\to-\sqrt{2}^+} \dfrac{x^2+1}{x^2-2} = -\infty, \quad \lim\limits_{x\to-\sqrt{2}^-} \dfrac{x^2+1}{x^2-2} = +\infty \end{array}\right\} \text{vertical asymptotes } x=\pm\sqrt{2}$$

$$\lim_{x\to\pm\infty} \frac{x^2+1}{x^2-2} = \lim_{x\to\pm\infty} \frac{1+\frac{1}{x^2}}{1-\frac{2}{x^2}} = 1$$: horizontal asymptote $Y=1$