Math 21A
Kouba
Inverse Functions and Derivatives

RECALL: If function $y=f(x)$ (graph passes vertical line test) is one-to-one (graph passes the horizontal line test), then f has an inverse function $y=f^{-1}(x)$ satisfying

$$\boxed{f(f^{-1}(x))=x}\ .$$

Notation: If $f(a)=b$, then $f^{-1}(b)=a$.

Example: (opt. prac.) Consider function $f(x)=\frac{x}{x-2}$ and note that $f(4)=2$.

1.) Verify algebraically that f is 1-1, i.e., if $f(x_1)=f(x_2)$, then $x_1=x_2$.

2.) Find the inverse function $y=f^{-1}(x)$.

3.) Compute $Df^{-1}(2)$.

Note: Sometimes finding $y=f^{-1}(x)$ is difficult or impossible. Is there a way to find $Df^{-1}(x)$ without first finding $y=f^{-1}(x)$?

__New Method__: If $f(f^{-1}(x)) = x$, then by Chain Rule

$$D\{f(f^{-1}(x))\} = D\{x\}$$

$$\rightarrow \quad f'(f^{-1}(x)) \cdot Df^{-1}(x) = 1$$

$$\rightarrow \quad \boxed{Df^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

__Example__: Consider function $f(x) = x^5 + x^3 + x + 5$ and note that $f(-1) = 2$.

1.) Show that f is 1-1 by using a derivative: $\xrightarrow{D}$

$f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all x-values $\rightarrow f$ is $\uparrow$ $\rightarrow f$ is 1-1.

2.) Find $Df^{-1}(2)$: We have $f^{-1}(2) = -1$ and

$$Df^{-1}(2) = \frac{1}{f'(f^{-1}(2))}$$

$$= \frac{1}{f'(-1)}$$

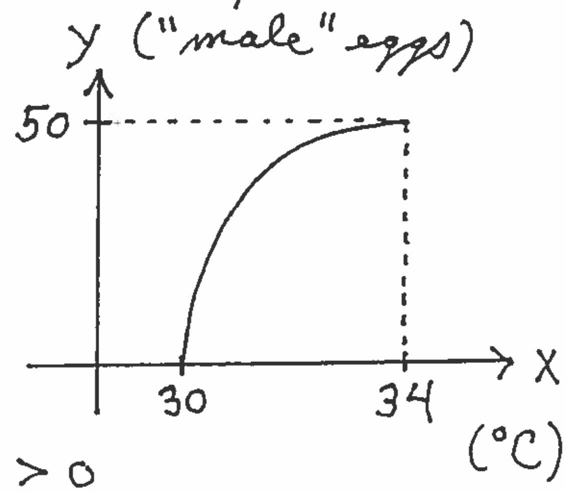
$$= \frac{1}{5(-1)^4 + 3(-1)^2 + 1} = \frac{1}{8}$$

Example: (opt. prac.) Consider function $f(x) = e^{-x} - 2x^3 + 5$ and note that $f(0) = 6$.

1.) Show that f is 1-1 by using a derivative.

2.) Find $D f^{-1}(6)$.

Example: Assume that function $f(x) = \frac{25}{34}\left[(x-34)^3 + (x-34)\right] + 50$ for $30 \le x \le 34$ is a mathematical model representing the number of "male" eggs in a clutch of 50 alligator eggs at temperature x (°C).



1.) Show that f is 1-1 and therefore has an inverse function $y = f^{-1}(x)$:

$\xrightarrow{D}$

$$f'(x) = \frac{25}{34}\left[3(x-34)^2 + 1\right] > 0$$

so f is $\uparrow$ $\rightarrow$ f is 1-1 $\rightarrow$ f^{-1} exists;

Note that

$f(30) = 0$ eggs, so $f^{-1}(0) = 30\,°C$
$f(31) \approx 28$ eggs, so $f^{-1}(28) \approx 31\,°C$
$f(32) \approx 42$ eggs, so $f^{-1}(42) \approx 32\,°C$
$f(33) \approx 48$ eggs, so $f^{-1}(48) \approx 33\,°C$
$f(34) = 50$ eggs, so $f^{-1}(50) = 34\,°C$;

the units for $Df(a)$ are $\dfrac{\text{eggs}}{°C}$,
the units for $Df^{-1}(b)$ are $\dfrac{°C}{\text{egg}}$;

2.) Find $Df(31)$:

$$Df(x) = \frac{25}{34}\left[3(x-34)^2+1\right] \rightarrow$$

$$Df(31) = \frac{25}{34}\left[3(31-34)^2+1\right] \approx 20.6 \ \frac{\text{eggs}}{°C}$$

3.) Find $Df^{-1}(28)$: note $f(31) = 28$ and $f^{-1}(28) = 31$

$$\rightarrow Df^{-1}(28) = \frac{1}{f'(f^{-1}(28))}$$

$$= \frac{1}{f'(31)}$$

$$\approx \frac{1}{20.6} \approx 0.05 \ \frac{°C}{\text{egg}}$$

4.) (opt. prac.) Find $Df(33)$ and $Df^{-1}(48)$.

5.) (opt. prac.) Find $Df(34)$ and $Df^{-1}(50)$.