

Math 21A (Summer Session 2006)
Kouba
Exam 3

KEY

Please PRINT your name here : -----

Your Exam ID Number -----

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.
2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION. VIOLATIONS CAN RESULT IN EXPULSION FROM THE UNIVERSITY.
3. No notes, books, or classmates may be used as resources for this exam. YOU MAY USE A CALCULATOR ON THIS EXAM.
4. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will receive little or no credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.
5. Make sure that you have 9 pages, including the cover page.
6. You will be graded on proper use of derivative and limit notation.
7. You have until 9:40 a.m. sharp to finish the exam.

1.) (5 pts. each) Use L'Hopital's Rule to evaluate the following limits.

$$\text{a.) } \lim_{x \rightarrow 0} \frac{\sin 6x}{\sin 3x} \stackrel{\text{"0/0"}}{=} \lim_{x \rightarrow 0} \frac{\cos 6x \cdot 6}{\cos 3x \cdot 3} = \frac{(1)(6)}{(1)(3)} = 2$$

$$\text{b.) } \lim_{x \rightarrow \infty} \frac{x^2}{5x + e^x} \stackrel{\text{"\infty/\infty"}}{=} \lim_{x \rightarrow \infty} \frac{2x}{5 + e^x}$$

$$\stackrel{\text{"\infty/\infty"}}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$$

$$\text{c.) } \lim_{x \rightarrow 0^+} x^2 \ln x = \text{"0} \cdot \infty \text{"} = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^2}}$$

$$\stackrel{\text{"\infty/\infty"}}{=} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x \rightarrow 0^+} -\frac{1}{2} x^2 = 0$$

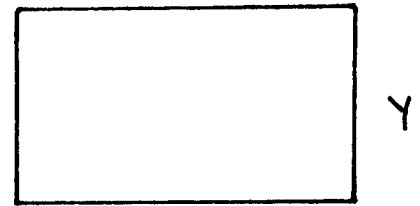
$$\text{d.) } \lim_{x \rightarrow \infty} (3 + 2x)^{1/x} \stackrel{\text{"\infty^0"}}{=} \lim_{x \rightarrow \infty} e^{\ln(3+2x)^{1/x}} = \lim_{x \rightarrow \infty} e^{\frac{1}{x} \ln(3+2x)}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{\ln(3+2x)}{x}} \stackrel{\text{"\infty/\infty"}}{=} e^{\lim_{x \rightarrow \infty} \frac{1}{3+2x} \cdot 2} = e^0 = 1.$$

2.) (7 pts.) Determine the dimensions of the rectangle of maximum area with a fixed perimeter of 20 feet.

perimeter $2x + 2y = 20$

$\rightarrow x + y = 10 \rightarrow \boxed{y = 10 - x}$;



max. area $A = xy = x(10 - x)$

$\rightarrow \boxed{A = 10x - x^2} \xrightarrow{D} A' = 10 - 2x = 0$



$x = 5 \text{ ft.}$

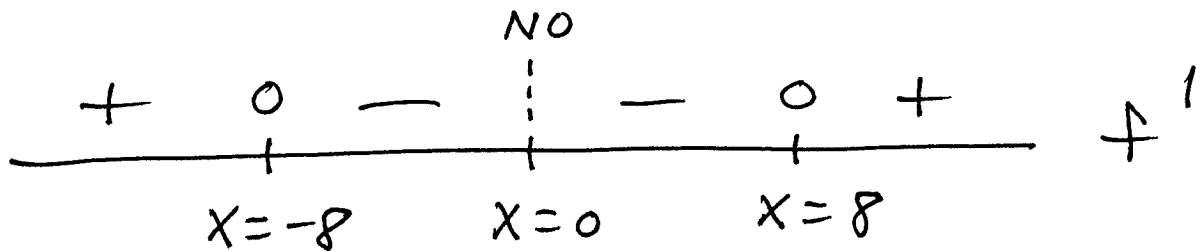
$y = 5 \text{ ft.}$

max. $A = 25 \text{ ft.}^2$

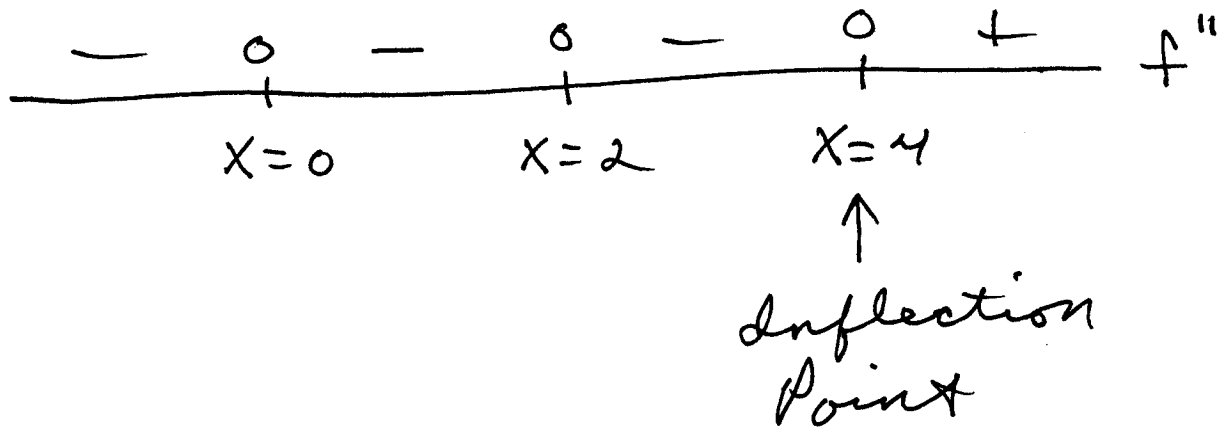
3.) a.) (7 pts.) Assume that $f(x) = x - 12 \cdot x^{1/3}$. Solve $f'(x) = 0$ for x and set up a sign chart for f' .

$$f'(x) = 1 - 12 \cdot \frac{1}{3} x^{-2/3} = 1 - \frac{4}{x^{2/3}} = \frac{x^{2/3} - 4}{x^{2/3}} = 0$$

$$(x^2)^{1/3} = 4 \rightarrow x^2 = 4^3 = 64 \rightarrow x = \pm 8$$



4.) (7 pts.) Assume that the second derivative of function f is $f''(x) = x^4(x-2)^2(x-4)^5$. Determine all x -values corresponding to inflection points for f .



5.) a.) (2 pts.) State the Mean Value Theorem (MVT). *assume f is cont. on $[a, b]$ and diff. on (a, b) . Then there is at least one c , $a < c < b$, so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

b.) (5 pts.) Determine if the following function satisfies the assumptions of the MVT on the given closed interval. If so, find all values of c guaranteed by the conclusion of the MVT.

$$f(x) = \sqrt{1-x} \text{ on } [0, 1]$$

$$f(x) = \sqrt{1-x} \text{ is cont.}$$

on $[0, 1]$ since it is the composition of cont. functions $y = \sqrt{x}$ and $y = 1-x$;

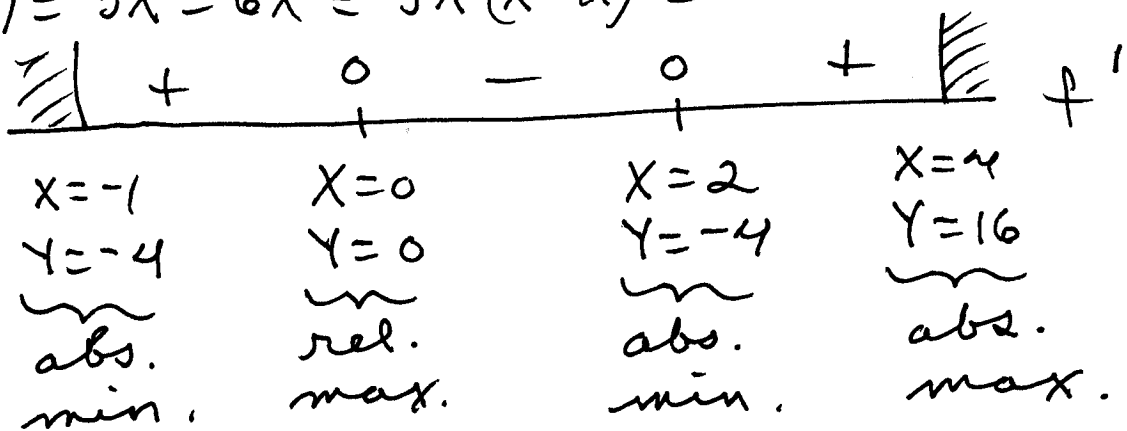
$$f'(x) = \frac{1}{2}(1-x)^{-1/2} \cdot (-1) = \frac{-1}{2\sqrt{1-x}} \text{ so } f \text{ is diff. on}$$

$$(0, 1). \text{ Then } f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{0 - 1}{1} = -1 \rightarrow$$

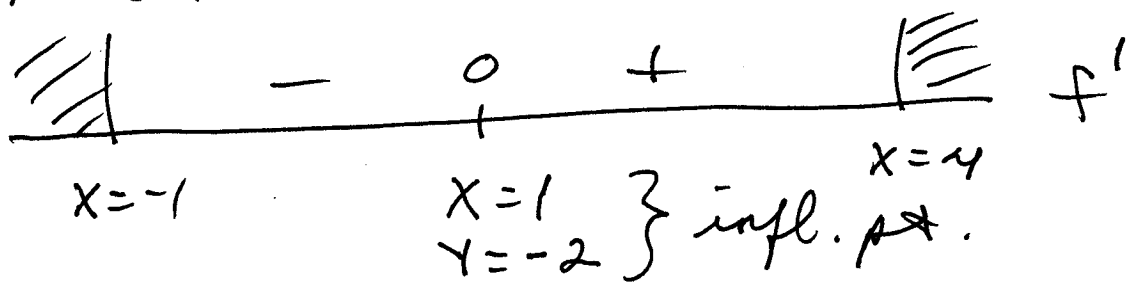
$$\frac{-1}{2\sqrt{1-c}} = -1 \rightarrow \frac{1}{2} = \sqrt{1-c} \rightarrow \frac{1}{4} = 1-c \rightarrow c = \frac{3}{4}$$

6.) (10 pts.) Consider the function $f(x) = x^2(x - 3)$ on the closed interval $[-1, 4]$. Determine where f is increasing (\uparrow), decreasing (\downarrow), concave up (\cup), and concave down (\cap). Identify all relative and absolute extrema, inflection points, and x - and y -intercepts. Sketch the graph. The first and second derivatives are $f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$.

$$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$$



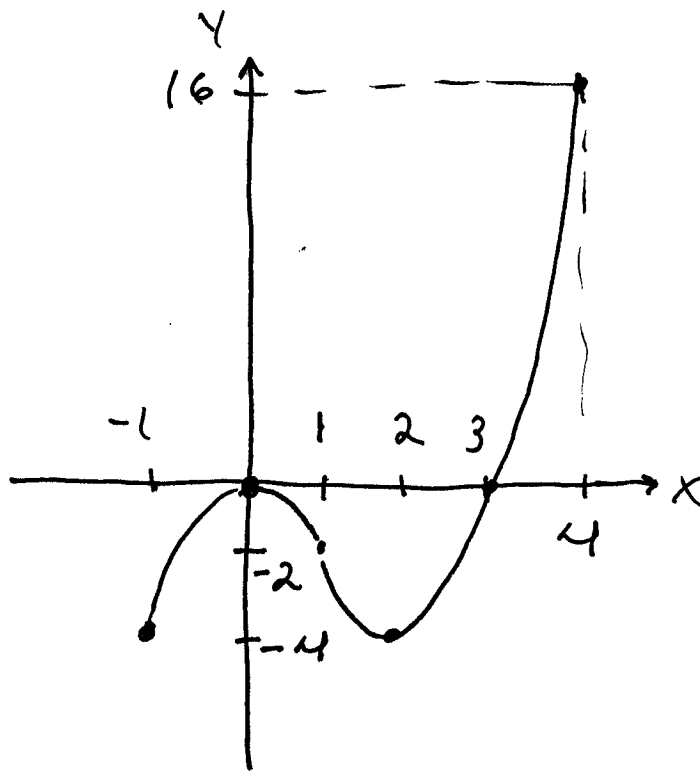
$$f''(x) = 6x - 6 = 6(x - 1) = 0$$



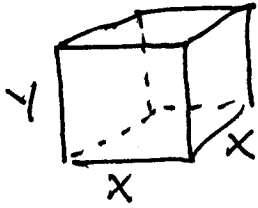
f is \uparrow for $-1 < x < 0, 2 < x < 4$,
 f is \downarrow for $0 < x < 2$,
 f is \cup for $1 < x < 4$,
 f is \cap for $-1 < x < 1$;

$$x = 0 : y = 0$$

$$y = 0 : x = 0, x = 3$$



7.) (7 pts.) A company wants to make a closed rectangular box with a square base and a fixed volume of 10 ft.^3 . If material for the top and bottom costs $\$5/\text{ft.}^2$ and the material for the sides costs $\$4/\text{ft.}^2$, find the dimensions and the cost of the least expensive box.



volume $x^2 y = 10 \text{ ft.}^3 \rightarrow \boxed{y = \frac{10}{x^2}}$;
 min. cost

$C = C_{\text{top}} + C_{\text{bot}} + C_{\text{sides}} \rightarrow$

$C = 5x^2 + 5x^2 + 4(4xy) = 10x^2 + 16xy \rightarrow$

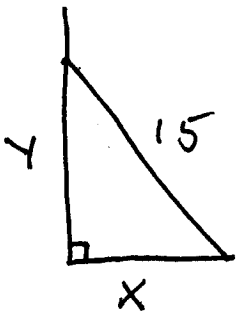
$C = 10x^2 + 16x\left(\frac{10}{x^2}\right) \rightarrow \boxed{C = 10x^2 + \frac{160}{x}} \xrightarrow{D}$

$C' = 20x - \frac{160}{x^2} = \frac{20x^3 - 160}{x^2} = \frac{20(x^3 - 8)}{x^2} = 0$

$\begin{array}{c} - & 0 & + \\ & | & \\ & x = 2 \text{ ft.} & \end{array} \quad C'$

$y = 5/2 \text{ ft.}, \text{ min. cost } C = \120

8.) (7 pts.) A 15-foot ladder is leaning against a wall. If the base of the ladder is pushed toward the wall at the rate of 2 ft./sec. , at what rate is the top of the ladder moving up the wall when the base of the ladder is 6 ft. from the wall?



assume $\frac{dx}{dt} = -2 \text{ ft./sec.}$; find

$\frac{dy}{dt}$ when $x = 6 \text{ ft.}$:

$x^2 + y^2 = 15^2 \xrightarrow{D} 2x \cdot \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$(6)(-2) + \sqrt{189} \cdot \frac{dy}{dt} = 0 \rightarrow$

$\frac{dy}{dt} = \frac{12}{\sqrt{189}} \approx 0.87 \text{ ft./sec.}$



$6^2 + y^2 = 15^2$
 $\rightarrow y = \sqrt{189}$
 ≈ 13.7

9.) (7 pts.) Use differentials to estimate the value of $\sqrt{96}$.

$$\text{Let } f(x) = \sqrt{x}, \quad x: 100 \rightarrow 96 \quad \text{so } \Delta x = -4,$$

$$f'(x) = \frac{1}{2\sqrt{x}}; \quad \Delta f = f(96) - f(100) = \sqrt{96} - 10,$$

$$df = f'(100) \cdot \Delta x = \frac{1}{20} \cdot (-4) = -\frac{1}{5}; \quad \text{assume}$$

$$\Delta f \approx df \rightarrow$$

$$\sqrt{96} - 10 \approx -\frac{1}{5} \rightarrow \sqrt{96} \approx 9.8$$

10.) (7 pts.) The radius and height of a cylinder are both equal to x so that the volume of the cylinder is given by $V = \pi x^3$. Assume that x is measured with an absolute percentage error of at most 3%. Use a differential to estimate the maximum absolute percentage error in computing the cylinder's volume.

Assume $\frac{|\Delta x|}{x} \leq 3\%$; find $\frac{|\Delta V|}{V}$:

$$V = \pi x^3 \rightarrow V' = 3\pi x^2, \quad \text{then}$$

$$\frac{|\Delta V|}{V} \approx \frac{|dV|}{V} = \frac{|V' \cdot \Delta x|}{V} = \frac{3\pi x^2 \cdot |\Delta x|}{\pi x^3}$$

$$= 3 \cdot \frac{|\Delta x|}{x} \leq 3(3\%) = 9\%$$

11.) (7 pts.) Determine the value of x in the given diagram which makes θ as large as possible.

$$\tan \alpha = \frac{3}{x} \rightarrow \alpha = \arctan\left(\frac{3}{x}\right);$$

$$\tan(\theta + \alpha) = \frac{8}{x} \rightarrow \theta + \alpha = \arctan\left(\frac{8}{x}\right)$$

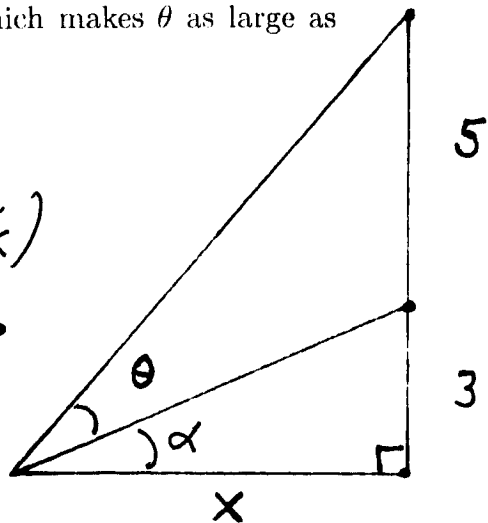
$$\rightarrow \theta = \arctan\left(\frac{8}{x}\right) - \arctan\left(\frac{3}{x}\right) \xrightarrow{D}$$

$$\theta' = \frac{1}{1 + \left(\frac{8}{x}\right)^2} \cdot \frac{-8}{x^2} - \frac{1}{1 + \left(\frac{3}{x}\right)^2} \cdot \frac{-3}{x^2}$$

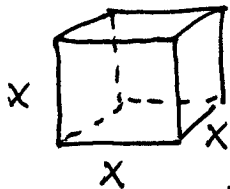
$$= \frac{-8}{x^2 + 64} + \frac{3}{x^2 + 9} = \frac{-8x^2 - 72 + 3x^2 + 192}{(x^2 + 64)(x^2 + 9)}$$

$$= \frac{-5x^2 + 120}{(x^2 + 64)(x^2 + 9)} = \frac{5(24 - x^2)}{(x^2 + 64)(x^2 + 9)} = 0 \dots \dots \dots \theta'$$

$$\dots \dots \dots \theta = \arctan\left(\frac{8}{\sqrt{24}}\right) - \arctan\left(\frac{3}{\sqrt{24}}\right) \approx 27.03^\circ$$



12.) (7 pts.) The surface area of a cube is increasing at the rate of $6 \text{ ft.}^2/\text{min}$. Determine the rate at which the volume of the cube is changing when the edge of the cube is 2 ft.



$$S = 6x^2, \quad V = x^3; \quad \text{assume } \frac{dS}{dt} = 6 \frac{\text{ft.}^2}{\text{min}};$$

find $\frac{dV}{dt}$ when $x = 2 \text{ ft.}$:

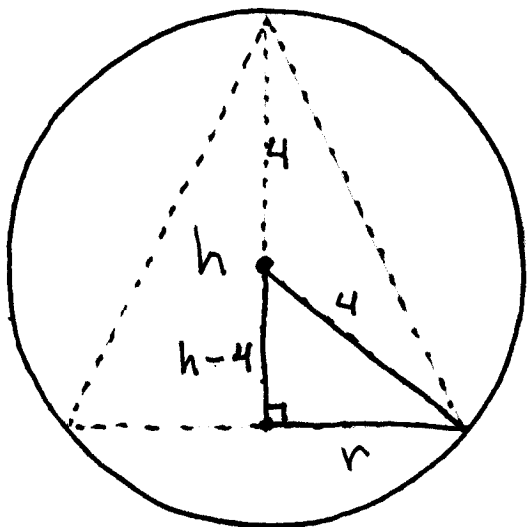
$$S = 6x^2 \xrightarrow{D} \frac{dS}{dt} = 12x \frac{dx}{dt} \rightarrow 6 = 12(2) \frac{dx}{dt} \rightarrow$$

$$\boxed{\frac{dx}{dt} = \frac{1}{4} \text{ ft./min}}; \quad V = x^3 \xrightarrow{D} \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$= 3(2)^2 \left(\frac{1}{4}\right) = 3 \text{ ft.}^3/\text{min.}$$

The following EXTRA CREDIT PROBLEM is worth 10 points. This problem is OPTIONAL.

1.) Determine the height h and the radius r of the cone of largest volume which can be inscribed in a sphere of radius 4 inches.



$$r^2 + (h-4)^2 = 16 \rightarrow$$

$$r^2 = 16 - (h^2 - 8h + 16) \rightarrow$$

$$\boxed{r^2 = 8h - h^2} ;$$

max. volume

$$V = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi (8h - h^2) \cdot h \rightarrow$$

$$\boxed{V = \frac{\pi}{3} (8h^2 - h^3)} \xrightarrow{D}$$

$$V' = \frac{\pi}{3} (16h - 3h^2) = \frac{\pi}{3} h (16 - 3h) = 0$$

$$\begin{array}{c} + \quad 0 \quad - \\ \hline \quad \quad | \quad \quad \\ h = \frac{16}{3} \text{ in.} \end{array} \quad V'$$

$$r = \sqrt{\frac{128}{9}} \approx 3.77 \text{ in.}$$

$$\text{max. } V = \frac{\pi}{3} \left(\frac{128}{9} \right) \left(\frac{16}{3} \right)$$

$$= \frac{2048}{81} \pi \approx 79.4 \text{ in.}^3$$