Math 21B (Kouba)
Comparison Tests for Improper Integrals

<u>Comparison Test for Convergence</u>

Assume f and g are continuous functions with $0 \le f(x) \le g(x)$. If $\int_a^\infty g(x)\,dx$ converges (finite), then $\int_a^\infty f(x)\,dx$ converges.

Ex: $$\int_2^\infty \frac{2x+1}{x^5+x^4+3}\,dx \le \int_2^\infty \frac{2x+x}{x^5+0+0}\,dx$$
$$= \int_2^\infty \frac{3x}{x^5}\,dx = 3\int_2^\infty \frac{1}{x^4}\,dx = 3 \cdot \lim_{A\to\infty} \int_2^A x^{-4}\,dx$$
$$= 3 \cdot \lim_{A\to\infty} \frac{-1}{3x^3}\Big|_2^A = 3 \cdot \lim_{A\to\infty} \left\{ \frac{-1}{3A^3} - \frac{-1}{24} \right\} = \frac{1}{24} < \infty,$$
so $\int_2^\infty \frac{2x+1}{x^5+x^4+3}\,dx$ converges.

<u>Comparison Test for Divergence</u>:

Assume f and g are continuous functions with $0 \le g(x) \le f(x)$. If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ diverges.

Ex: $$\int_1^\infty \frac{x^3+3}{x^4+1}\,dx \ge \int_1^\infty \frac{x^3+0}{x^4+x^4}\,dx$$
$$= \int_1^\infty \frac{x^3}{2x^4}\,dx = \frac{1}{2}\int_1^\infty \frac{1}{x}\,dx = \frac{1}{2}\lim_{A\to\infty}\int_1^A \frac{1}{x}\,dx$$
$$= \frac{1}{2}\lim_{A\to\infty} \ln|x|\Big|_1^A = \frac{1}{2}\lim_{A\to\infty}(\ln A - \underbrace{\ln 1}_{0}) = \infty,$$
so $\int_1^\infty \frac{x^3+3}{x^4+1}\,dx$ diverges.