Section 6.3

1.) $\text{Arc} = \int_{-\frac{2}{3}}^{1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_{-\frac{2}{3}}^{1} \sqrt{(-1)^2 + (3)^2}\, dt = \int_{-2/3}^{1} \sqrt{10}\, dt$

$= \sqrt{10}\, t \Big|_{-2/3}^{1} = \sqrt{10}\left(1 - -\frac{2}{3}\right) = \frac{5}{3}\sqrt{10}$

2.) $\text{Arc} = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int_0^{\pi} \sqrt{(-\sin t)^2 + (1+\cos t)^2}\, dt$

$= \int_0^{\pi} \sqrt{\underline{\sin^2 t + \cos^2 t} + 2\cos t + 1}\, dt$

$= \int_0^{\pi} \sqrt{2 + 2\cos t}\, dt$

$= \int_0^{\pi} \sqrt{2(1+\cos t)\frac{(1-\cos t)}{(1-\cos t)}}\, dt$

$= \int_0^{\pi} \sqrt{2} \cdot \sqrt{\frac{1-\cos^2 t}{1-\cos t}}\, dt$

$= \sqrt{2} \int_0^{\pi} \frac{\sqrt{\sin^2 t}}{\sqrt{1-\cos t}}\, dt$

$= \sqrt{2} \int_0^{\pi} \frac{|\sin t|}{\sqrt{1-\cos t}}\, dt$

$= \sqrt{2} \int_0^{\pi} \frac{\sin t}{\sqrt{1-\cos t}}\, dt$

(Let $u = 1 - \cos t \rightarrow du = \sin t\, dt$;
$t : 0 \rightarrow \pi$ so $u : 0 \rightarrow 2$)

$$= \sqrt{2} \int_0^2 \frac{1}{\sqrt{u}}\, du = \sqrt{2} \cdot \int_0^2 u^{-1/2}\, du$$

$$= \sqrt{2} \cdot \frac{u^{1/2}}{1/2} \Big|_0^2 = 2\sqrt{2} \cdot \sqrt{2} = 4$$

4.) $$\text{Arc} = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^4 \sqrt{(t)^2 + \left(\frac{3}{\cancel{2}} \cdot \frac{1}{\cancel{3}} (2t+1)^{1/2} \cdot \cancel{2}\right)^2}\, dt$$

$$= \int_0^4 \sqrt{t^2 + 2t + 1}\, dt$$

$$= \int_0^4 \sqrt{(t+1)^2}\, dt = \int_0^4 (t+1)\, dt$$

$$= \left(\frac{1}{2}t^2 + t\right)\Big|_0^4 = 8 + 4 = 12$$

6.) $$\text{Arc} = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\begin{aligned}&(\cancel{-8\sin t} + 8t\cos t + \cancel{8\sin t})^2 \\ &+ (\cancel{8\cos t} - (8t \cdot -\sin t + \cancel{8\cos t}))^2\end{aligned}}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(8t\cos t)^2 + (8t\sin t)^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{64t^2\cos^2 t + 64t^2\sin^2 t}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{64t^2 (\underbrace{\cos^2 t + \sin^2 t}_{1})}\, dt$$

$$= \int_0^{\frac{\pi}{2}} 8t\, dt = 4t^2 \Big|_0^{\frac{\pi}{2}}$$

$$= 4\left(\frac{\pi}{2}\right)^2 = \pi^2$$

8.) $$\text{Arc} = \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt$$

$$= \int_0^{\pi} \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2}\; dt$$

$$= \int_0^{\pi} \sqrt{(e^t)^2(\cos t - \sin t)^2 + (e^t)^2(\cos t + \sin t)^2}\; dt$$

$$= \int_0^{\pi} \sqrt{(e^t)^2} \cdot \sqrt{\cos^2 t - \cancel{2\sin t \cos t} + \sin^2 t + \cos^2 t + \cancel{2 \sin t \cos t} + \sin^2 t}\; dt$$

$$= \int_0^{\pi} e^t \sqrt{1+1}\; dt = \sqrt{2}\, e^t \Big|_0^{\pi}$$

$$= \sqrt{2}\,(e^{\pi} - e^0) = \sqrt{2}\,(e^{\pi} - 1)$$

9.) $$\text{Arc} = \int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx = \int_0^3 \sqrt{1 + \left(\frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}} (x^2+2)^{1/2} \cdot \cancel{2}x\right)^2}\; dx$$

$$= \int_0^3 \sqrt{1 + (x^2+2)\,x^2}\; dx = \int_0^3 \sqrt{x^4 + 2x^2 + 1}\; dx$$

$$= \int_0^3 \sqrt{(x^2+1)^2}\; dx = \int_0^3 (x^2+1)\, dx$$

$$= \left(\frac{x^3}{3} + x\right)\Big|_0^3 = 9 + 3 = 12$$

10.) $$\text{Arc} = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx = \int_0^4 \sqrt{1 + \left(\frac{3}{2} x^{1/2}\right)^2}\; dx$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\; dx = \frac{4}{9}\, \frac{\left(1 + \frac{9}{4}x\right)^{3/2}}{3/2}\Big|_0^4$$

$$= \frac{8}{27}\left(10^{3/2} - 1^{3/2}\right) = \frac{8}{27}\left(10^{3/2} - 1\right)$$

11.) $$\text{Arc} = \int_1^3 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\; dy$$

$$= \int_1^3 \sqrt{1 + \left(y^2 - \frac{1}{4y^2}\right)^2}\, dy$$

$$= \int_1^3 \sqrt{1 + y^4 - \frac{1}{2} + \frac{1}{16y^4}}\, dy$$

$$= \int_1^3 \sqrt{y^4 + \frac{1}{2} + \frac{1}{16y^4}}\, dy$$

$$= \int_1^3 \sqrt{\frac{16y^8}{16y^4} + \frac{8y^4}{16y^4} + \frac{1}{16y^4}}\, dy$$

$$= \int_1^3 \sqrt{\frac{16y^8 + 8y^4 + 1}{16y^4}}\, dy$$

$$= \int_1^3 \frac{\sqrt{(4y^4+1)^2}}{4y^2}\, dy$$

$$= \int_1^3 \frac{4y^4+1}{4y^2}\, dy$$

$$= \int_1^3 \left(y^2 + \frac{1}{4}y^{-2}\right) dy$$

$$= \left(\frac{y^3}{3} - \frac{1}{4y}\right)\Big|_1^3$$

$$= \left(9 - \frac{1}{12}\right) - \left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= 9 - \frac{1}{6} = \frac{53}{6}$$

12.) $\text{Arc} = \int_1^9 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$$= \int_1^9 \sqrt{1 + \left(\frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2} y^{1/2} - \frac{1}{2} y^{-1/2}\right)^2}\, dy$$

$$= \int_1^9 \sqrt{1 + \frac{y}{4} - \frac{1}{2} + \frac{1}{4y}}\ dy$$

$$= \int_1^9 \sqrt{\frac{y}{4} + \frac{1}{2} + \frac{1}{4y}}\ dy$$

$$= \int_1^9 \sqrt{\frac{y^2}{4y} + \frac{2y}{4y} + \frac{1}{4y}}\ dy$$

$$= \int_1^9 \sqrt{\frac{y^2+2y+1}{4y}}\ dy$$

$$= \int_1^9 \frac{\sqrt{(y+1)^2}}{2\sqrt{y}}\ dy$$

$$= \int_1^9 \frac{y+1}{2\sqrt{y}}\ dy = \int_1^9 \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right) dy$$

$$= \left(\frac{1}{\cancel{2}}\frac{y^{3/2}}{3/\cancel{2}} + \frac{1}{\cancel{2}}\cdot\frac{y^{1/2}}{1/\cancel{2}}\right)\Big|_1^9$$

$$= \left(\frac{1}{3}9^{3/2} + 3\right) - \left(\frac{1}{3} + 1\right)$$

$$= 12 - \frac{4}{3} = \frac{32}{3}$$

17.) $\text{Arc} = \int_{-1/2}^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$

$$= \int_{-1/2}^{1/2} \sqrt{1 + \left(\frac{1}{\cancel{2}}(1-x^2)^{-1/2}\cdot(-\cancel{2}x)\right)^2}\ dx$$

$$= \int_{-1/2}^{1/2} \sqrt{1 + \frac{x^2}{1-x^2}}\ dx$$

$$= \int_{-1/2}^{1/2} \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}}\ dx$$

$$= \int_{-1/2}^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \Big|_{-1/2}^{1/2}$$

$$= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(\frac{-1}{2}\right)$$

$$= \frac{\pi}{6} - \frac{-\pi}{6} = \frac{\pi}{3}$$

29.) a.) $L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\,dx = \int_1^4 \sqrt{1 + (f'(x))^2}\,dx$

$\rightarrow (f'(x))^2 = \frac{1}{4x} \longrightarrow f'(x) = \sqrt{\frac{1}{4x}} \longrightarrow$

$f'(x) = \frac{1}{2\sqrt{x}} \longrightarrow f(x) = \sqrt{x} + C$

and $x = 1,\ y = 1 \longrightarrow 1 = \sqrt{1} + C \rightarrow C = 0 \rightarrow$

$f(x) = \sqrt{x}$.

30.) a.) $L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\,dy = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$

$\longrightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{y^4} \longrightarrow \frac{dx}{dy} = \frac{1}{y^2} \longrightarrow$

$x = \frac{-1}{y} + C$ and $x = 0,\ y = 1 \longrightarrow$

$0 = -1 + C \rightarrow C = 1 \longrightarrow x = \frac{-1}{y} + 1$.

35.) $$\text{Arc} = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^3 \sqrt{(e^t - e^{-t})^2 + (-2)^2}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} - 2 + e^{-2t} + 4}\, dt$$

$$= \int_0^3 \sqrt{e^{2t} + 2 + e^{-2t}}\, dt$$

$$= \int_0^3 \sqrt{(e^t + e^{-t})^2}\, dt$$

$$= \int_0^3 (e^t + e^{-t})\, dt = (e^t - e^{-t})\Big|_0^3$$

$$= (e^3 - e^{-3}) - \overset{0}{(e^0 \cancel{-} e^0)}$$

$$= e^3 - \frac{1}{e^3}$$