Math 21B
Kouba
Recursion (Reduction) Formulas

Reduction (Recursion) formulas are used to write a difficult integral in simpler form in successive steps. The following example illustrates this idea.

EXAMPLE of RECURSION:

$$\int \sin^n x \ dx = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \ dx$$

HOW TO USE IT:

$$\int \sin^4 x \ dx \quad (Use\ n=4) \quad = -\frac{\sin^3 x \cdot \cos x}{4} + \frac{3}{4}\int \sin^2 x \ dx \quad (Use\ n=2)$$

$$= -\frac{\sin^3 x \cdot \cos x}{4} + \frac{3}{4}\left[-\frac{\sin x \cdot \cos x}{2} + \frac{1}{2}\int \sin^0 x \ dx\right]$$

$$= -\frac{\sin^3 x \cdot \cos x}{4} - \frac{3}{8}\sin x \cdot \cos x + \frac{3}{8}\int 1 \ dx$$

$$= -\frac{\sin^3 x \cdot \cos x}{4} - \frac{3}{8}\sin x \cdot \cos x + \frac{3}{8}\ x$$

DERIVE THE RECURSION:

$$\int \sin^n x \ dx = \int \sin^{n-2} x \cdot \sin^2 x \ dx$$

$$= \int \sin^{n-2} x \cdot (1 - \cos^2 x) \ dx$$

$$= \int \sin^{n-2} x \ dx - \int (\sin^{n-2} x \cdot \cos x) \cos x \ dx$$

(Let $u = \cos x$ and $dv = \sin^{n-2} x \cdot \cos x \ dx$,

then $du = -\sin x \ dx$ and $v = \dfrac{1}{n-1}\sin^{n-1} x$)

$$= \int \sin^{n-2} x \ dx - \left[\frac{\sin^{n-1} x \cdot \cos x}{n-1} - -\int \frac{\sin x \cdot \sin^{n-1} x}{n-1} \ dx\right]$$

$$= \int \sin^{n-2} x \ dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1} - \frac{1}{n-1}\int \sin^n x \ dx \quad \text{(Now the TWIST !)}$$

$$\longrightarrow \int \sin^n x \; dx = \int \sin^{n-2} x \; dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1} - \frac{1}{n-1} \int \sin^n x \; dx$$

$$\longrightarrow \left(1 + \frac{1}{n-1}\right) \int \sin^n x \; dx = \int \sin^{n-2} x \; dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1}$$

$$\longrightarrow \frac{n}{n-1} \int \sin^n x \; dx = \int \sin^{n-2} x \; dx - \frac{\sin^{n-1} x \cdot \cos x}{n-1}$$

$$\longrightarrow \int \sin^n x \; dx = -\frac{\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \; dx$$

EXAMPLE: (Optional Practice) Derive each of the following recursions.

1.) $\displaystyle\int \tan^n x \; dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \; dx$

2.) $\displaystyle\int \cos^n x \; dx = \frac{\cos^{n-1} x \cdot \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \; dx$

EXAMPLE: (Optional Practice) Use recursions to do the following integrals.

1.) $\displaystyle\int \tan^3 x \; dx$

2.) $\displaystyle\int \cos^6 x \; dx$