Derivation of Trapezoidal Rule Error Formula

$$|E_n| = \left| \int_a^b f(x)\,dx - T_n \right| \le (b-a)\frac{h^2}{12}\left\{\max_{a \le x \le b} |f''(x)|\right\},$$

where $h = \frac{b-a}{n}$ and

$$T_n = \frac{h}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]:$$

Without loss of generality, assume $a = 0$ and $h = \frac{b-0}{n} = \frac{b}{n}$, so that

$$\int_0^b f(x)\,dx = \int_0^h f(x)\,dx + \int_h^{2h} f(x)\,dx + \cdots + \int_{b-h}^b f(x)\,dx .$$

Start with $\int_0^h f(x)\,dx$ and integrate by parts twice;

(Let $u = f(x)$, $dv = dx$, $du = f'(x)\,dx$, $v = x + A$)

$$\int_0^h f(x)\,dx = (x+A)f(x)\Big|_0^h - \int_0^h (x+A)f'(x)\,dx$$

(Let $u = f'(x)$, $dv = (x+A)\,dx$, $du = f''(x)\,dx$, $v = \frac{1}{2}(x+A)^2 + B$), then

$$(*)\ \int_0^h f(x)\,dx = (x+A)f(x)\Big|_0^h - \left[\left(\tfrac{1}{2}(x+A)^2 + B\right)f'(x)\Big|_0^h - \int_0^h \left(\tfrac{1}{2}(x+A)^2 + B\right) f''(x)\,dx\right].$$

Now choose integration constants A and B so that

$$\text{I.)} \quad (x+A)\,f(x)\Big|_0^h = \frac{h}{2}\left[f(0)+f(h)\right]$$

and

$$\text{II.)} \quad \left(\tfrac{1}{2}(x+A)^2+B\right)f'(x)\Big|_0^h = 0 \quad .$$ Then

$$(h+A)\,f(h)-(0+A)\,f(0) = \frac{h}{2}\,f(0)+\frac{h}{2}\,f(h) \rightarrow$$

$$h\,f(h)+A\,f(h)-A\,f(0) = \frac{h}{2}\,f(0)+\frac{h}{2}\,f(h) \rightarrow$$

$$\left(A+\frac{h}{2}\right)f(h)-\left(A+\frac{h}{2}\right)f(0) = 0 ,$$ and

$\boxed{A = \frac{-h}{2}}$ works. Substitute into II.) getting

$$\left(\tfrac{1}{2}\left(x-\frac{h}{2}\right)^2+B\right)f'(x)\Big|_0^h = 0 \rightarrow$$

$$\left(\frac{h^2}{8}+B\right)f'(h)-\left(\frac{h^2}{8}+B\right)f'(0) = 0 ,$$

and $\boxed{B = -\frac{h^2}{8}}$ works. Now equation (*) becomes

$$(*)(*) \quad \int_0^h f(x)\,dx = \frac{h}{2}\left(f(0)+f(h)\right)+\int_0^h\left(\tfrac{1}{2}\left(x-\tfrac{h}{2}\right)^2-\tfrac{h^2}{8}\right)f''(x)\,dx$$

$$= \frac{h}{2}\left(f(0)+f(h)\right)+\int_0^h \tfrac{1}{2}\left(x^2-hx\right)f''(x)\,dx .$$

Theorem: If f and g are continuous on $[a,b]$, then

$$\int_a^b f(x)g(x)\,dx = g(c)\int_a^b f(x)\,dx$$

for some c in $[a,b]$.

Proof: Since g is continuous on $[a,b]$, g has a minimum value $g(\alpha)=m$ and a maximum value $g(\beta)=M$. Thus, (assume $f(x)\geq 0$)

$$g(\alpha)\leq g(x)\leq g(\beta) \rightarrow g(\alpha)f(x)\leq f(x)g(x)\leq g(\beta)f(x)$$

$$\rightarrow \text{III.)}\quad g(\alpha)\int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq g(\beta)\int_a^b f(x)\,dx.$$

Define function $K=K(t)$ on $[a,b]$ by $K(t)=g(t)\int_a^b f(x)\,dx$. From III.) we have that

$$K(\alpha)\leq \int_a^b f(x)g(x)\,dx \leq K(\beta).$$

Since K is continuous it follows from the Intermediate Value Theorem that

$$K(c)=\int_a^b f(x)g(x)\,dx$$

for some c in $[a,b]$, i.e.,

$$g(c)\int_a^b f(x)\,dx = \int_a^b f(x)g(x)\,dx.$$

Q.E.D.

applying this theorem to (*)/(*) we get

$$E_1^* = \int_0^h f(x)\,dx - \frac{h}{2}(f(0)+f(h))$$
$$= \int_0^h \frac{1}{2}(x^2 - hx) f''(x)\,dx$$
$$= f''(c_1) \int_0^h \frac{1}{2}(x^2 - hx)\,dx$$

for some c_1 in $[a,b]$

$$= f''(c_1) \cdot \frac{1}{2}\left(\frac{1}{3}x^3 - \frac{h}{2}x^2\right)\Big|_0^h$$
$$= \frac{-h^3}{12} \cdot f''(c_1) \quad .$$

Similarly,

$$E_2^* = \int_h^{2h} f(x)\,dx - \frac{h}{2}(f(h) + f(2h))$$
$$= \frac{-h^3}{12} \cdot f''(c_2) \; ,$$
$$E_3^* = \int_{2h}^{3h} f(x)\,dx - \frac{h}{2}(f(2h) + f(3h))$$
$$= \frac{-h^3}{12} \cdot f''(c_3) \; , \; \cdots \; ,$$
$$E_n^* = \int_{b-h}^{b} f(x)\,dx - \frac{h}{2}(f(b-h) + f(b))$$
$$= \frac{-h^3}{12} \cdot f''(c_n) \quad .$$

combining these n results gives

$$E_n = \int_a^b f(x)\,dx - T_n$$

$$= \int_0^b f(x)\,dx - \frac{h}{2}[f(0) + 2f(h) + \cdots + 2f(b-h) + f(b)]$$

$$= \int_0^h f(x)\,dx - \frac{h}{2}(f(0) + f(h))$$

$$+ \int_h^{2h} f(x)\,dx - \frac{h}{2}(f(h) + f(2h))$$

$$+ \int_{2h}^{3h} f(x)\,dx - \frac{h}{2}(f(2h) + f(3h))$$

$$+ \cdots$$

$$+ \int_{b-h}^{b} f(x)\,dx - \frac{h}{2}(f(b-h) + f(b))$$

$$= E_1^* + E_2^* + E_3^* + \cdots + E_n^*$$

$$= \frac{-h^3}{12} f''(c_1) + \frac{-h^3}{12} f''(c_2) + \cdots + \frac{-h^3}{12} f''(c_n)$$

$$= -\frac{h^2}{12} \cdot \frac{b-0}{n} (f''(c_1) + f''(c_2) + \cdots + f''(c_n))$$

$$(\#) \quad = -(b-0)\frac{h^2}{12}\left(\frac{f''(c_1) + f''(c_2) + \cdots + f''(c_n)}{n}\right).$$

Since f is continuous on $[a, b]$, f has a minimum value $f(\alpha) = m$ and a maximum value $f(\beta) = M$.

It follows that

$$n f(\alpha) \le f(c_1) + f(c_2) + \cdots + f(c_n) \le n f(\beta) \rightarrow$$

$$f(\alpha) \le \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} \le f(\beta) .$$

By the Intermediate Value Theorem

$$f(c) = \frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n}$$

for some c in $[a, b]$. Thus (#) becomes

$$E_n = \int_0^b f(x)\,dx - T_n$$

$$= -(b-0)\frac{h^2}{12} f''(c) \text{, so that}$$

$$|E_n| = \left| \int_0^b f(x)\,dx - T_n \right|$$

$$= (b-0) \cdot \frac{h^2}{12} \cdot |f''(c)|$$

$$\le (b-0)\frac{h^2}{12} \left\{ \max_{0 \le x \le b} |f''(x)| \right\}$$