Math 21C
Kouba
Complex Numbers

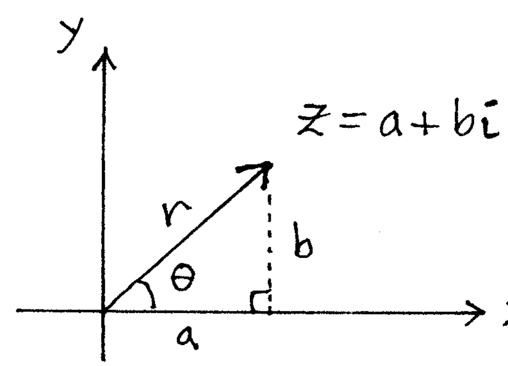
Def: $i = \sqrt{-1}$.

Then $i^2 = (\sqrt{-1})^2 = -1$,
$i^3 = i^2 \cdot i = -i$,
$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1$,
$i^5 = i^4 \cdot i = (1)i = i$,
$i^6 = i^4 \cdot i^2 = (1)(-1) = -1$,
$i^7 = i^4 \cdot i^3 = (1)(-i) = -i$,
$i^8 = i^4 \cdot i^4 = (1)(1) = 1$, ...

Ex: $i^{83} = (i^4)^{20} \cdot i^3 = (1)^{20}(-i) = -i$
Ex: Solve $x^2+1=0$. Then
$x^2 = -1 \Rightarrow x = i$ or $x = -i$.

Def: a complex number is any number of the form $z = a+bi$, where a and b are real numbers.

Ex: $3-i$, $2+7i$, $-4i$, $7/2$ are all complex numbers.



Note: The complex number $z = a+bi$ can be represented in the xy-plane as a vector from the origin with length r and direction θ. From polar coordinates we know that

$$\begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases} \quad \text{and} \quad r = \sqrt{a^2+b^2}. \text{ Then}$$

$$\boxed{z = a+bi} \quad \text{(rectangular form)}$$

and

$$z = r\cos\theta + r\sin\theta \cdot i \Rightarrow$$

$$\boxed{z = r(\cos\theta + i\sin\theta)} \quad \text{(polar form)}.$$

Def: The magnitude (length) of $z = a+bi$ is $|z| = \sqrt{a^2+b^2}$. The argument (direction) of $z = a+bi$ is θ, where $\tan\theta = b/a$.

Note: 1.) If $z = \cos\theta + i\sin\theta$, then

$$|z| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1.$$

2.) If $z = a+bi$, then $|z| = \sqrt{a^2+b^2}$ and

$$w = \frac{z}{|z|} = \frac{1}{\sqrt{a^2+b^2}}(a+bi) = \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}}i$$

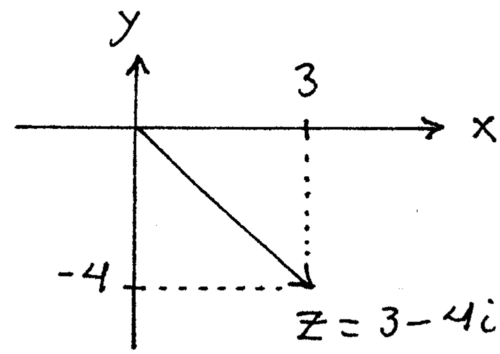
has the same direction as z and $|w| = 1$; if we let $\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}}$ and $\sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$, then $w = \cos\theta + i\sin\theta$.

Ex: Simplify each and write answers in the form $z = a+bi$; plot z and compute $|z|$.

1.) $(2+i)+(1-5i)$:

$(2+i)+(1-5i) = 3-4i$;

$|3-4i| = \sqrt{3^2+(-4)^2} = \sqrt{25} = 5$

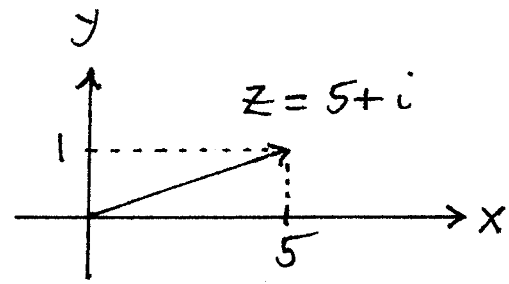


2.) $(3-2i)(1+i)$:

$(3-2i)(1+i) = 3+3i-2i-2i^2$

$= 3+i-2(-1) = 5+i$;

$|5+i| = \sqrt{5^2+1^2} = \sqrt{26}$

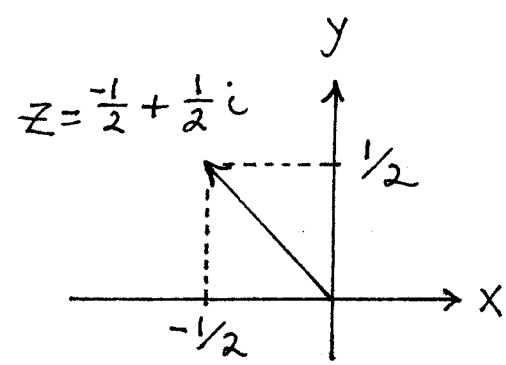


3.) $\dfrac{i}{1-i}$:

$$\frac{i}{1-i} = \frac{i}{1-i}\cdot\frac{1+i}{1+i} = \frac{i+i^2}{1^2-i^2}$$

$$= \frac{i-1}{1-(-1)} = \frac{i-1}{2} = \frac{-1}{2}+\frac{1}{2}i;$$

$$\left|\frac{-1}{2}+\frac{1}{2}i\right| = \sqrt{\left(\frac{-1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

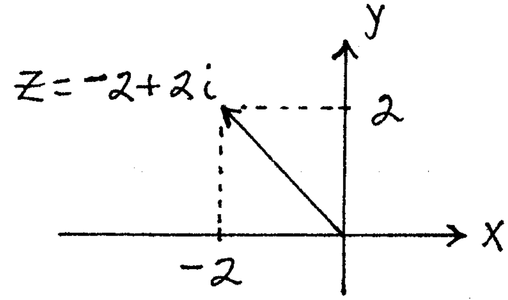


4.) $(1+i)^3$:

$(1+i)^3 = 1^3+3i+3i^2+i^3$

$= 1+3i-3-i = -2+2i$;

$|-2+2i| = \sqrt{(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}$



5.) $(1+i)^{20}$?! * (need new tools)

<u>Theorem</u>: Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$. Then the magnitude of zw is $|zw| = r_1 r_2$ and the argument

is argument $(zw) = \theta_1 + \theta_2$.
(multiply lengths, add angles)

Proof: $zw = r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)$

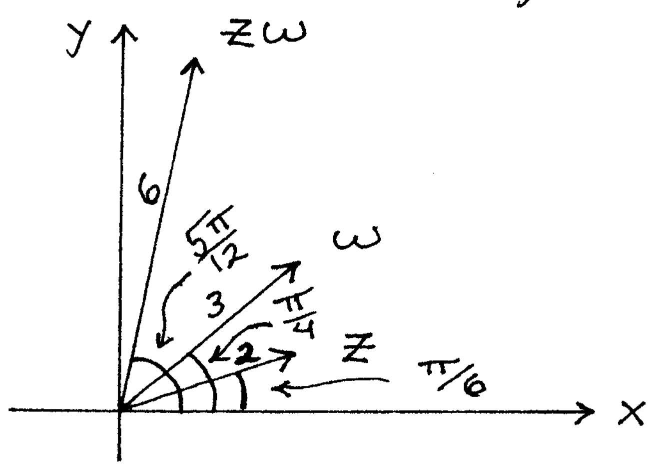
$$= r_1 r_2 [\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2]$$

$$= r_1 r_2 [(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Fact: If $z = r(\cos\theta + i\sin\theta)$, then
$z^n = r^n(\cos\theta + i\sin\theta)^n = r^n(\cos n\theta + i\sin n\theta)$.
(De Moivre's law)

Ex: If $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and
$w = 3(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, then
$zw = 6(\cos(\frac{\pi}{6} + \frac{\pi}{4}) + i\sin(\frac{\pi}{6} + \frac{\pi}{4}))$
$= 6(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12})$.



Ex: Let $z = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$. Compute and plot z, z^2, z^3, z^4, and z^5. Write answers in rectangular form.

$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = \sqrt{3} + i$;

$z^2 = 2^2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^2 = 4\left(\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}\right)$

$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} + i\cdot\frac{\sqrt{3}}{2}\right) = 2 + 2\sqrt{3}\,i$;

$z^3 = 2^3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3 = 8\left(\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6}\right)$

$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 8(0 + i) = 8i$;

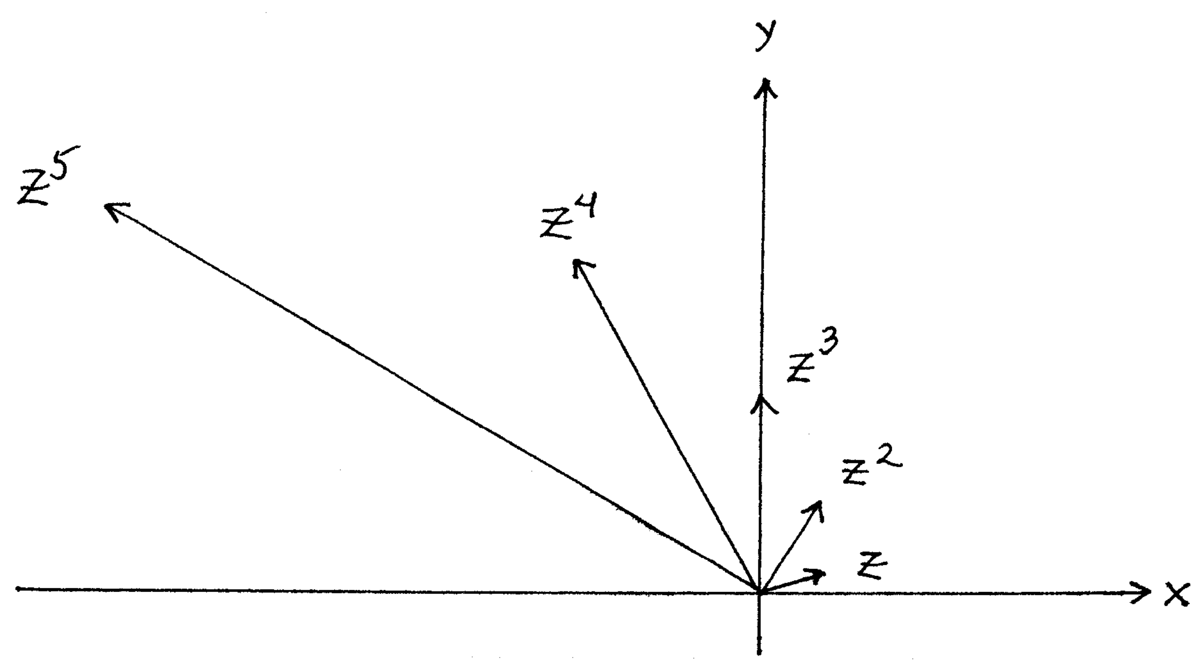
$z^4 = 2^4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^4 = 16\left(\cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}\right)$

$= 16\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 16\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) = -8 + 8\sqrt{3}\,i$;

$z^5 = 2^5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5 = 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 32\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right) = -16\sqrt{3} + 16i$; ...

$z^n = 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n = 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ has magnitude 2^n and argument $\frac{n\pi}{6}$.



<u>Ex</u>: Simplify $(1+i)^{20}$ and write answer in rectangular form.

Let $z = 1+i$, then $|z| = \sqrt{1^2+1^2} = \sqrt{2} \Rightarrow$

$$z = 1+i = \sqrt{2} \cdot \frac{1+i}{\sqrt{2}}$$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right)$$

$$= \sqrt{2} \left(\cos\theta + i \sin\theta \right)$$

$$= \sqrt{2} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right) \quad ;$$

then

$$(1+i)^{20} = \left(\sqrt{2} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right) \right)^{20}$$

$$= \left(\sqrt{2}\right)^{20} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)^{20}$$

$$= 2^{10} \left(\cos\frac{20\pi}{4} + i \sin\frac{20\pi}{4} \right)$$

$$= 1024 \left(\cos 5\pi + i \sin 5\pi \right)$$

$$= 1024 \left(-1 + i \cdot (0) \right)$$

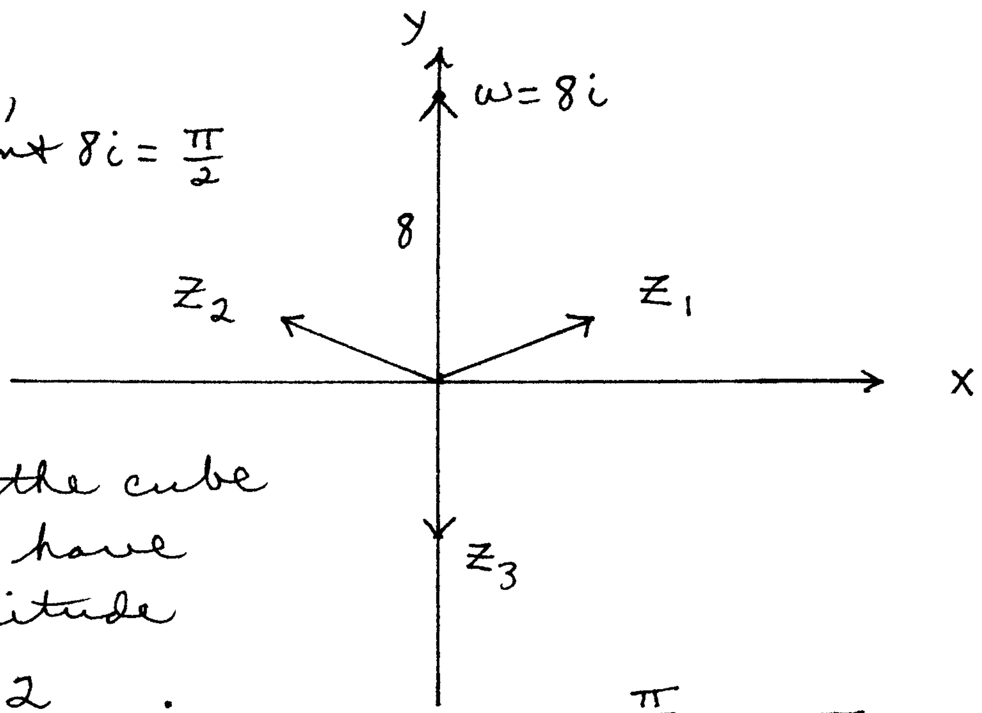
$$= -1024 \quad .$$

<u>Fact</u>: Let $w = a + bi$ be a fixed complex number. Then the equation $z^n = w$ has n equally-spaced solutions of equal length, where n is a positive integer.

<u>Ex</u>: Solve $z^3 = 8i$ for z, i.e., find all three cube roots of $8i$:

$|8i| = 8$,

argument $8i = \frac{\pi}{2}$



All of the cube roots have magnitude $8^{1/3} = 2$.

The argument of $z_1 = \dfrac{\frac{\pi}{2}}{3} = \dfrac{\pi}{6}$. The other cube roots are equally spaced, so there is an angle of $\frac{2\pi}{3}$ between consecutive roots:

$$z_1 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$
$$= 2\left(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = \sqrt{3} + i \;;$$

$$z_2 = 2\left(\cos\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)+i\ \sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right)$$
$$= 2\left(\cos\frac{5\pi}{6} + i\ \sin\ \frac{5\pi}{6}\right)$$
$$= 2\left(\frac{-\sqrt{3}}{2} + i\cdot\frac{1}{2}\right) = -\sqrt{3}+i\ ;$$
$$z_3 = 2\left(\cos\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)+i \sin\left(\frac{\pi}{6}+\frac{4\pi}{3}\right)\right)$$
$$= 2\left(\cos\frac{3\pi}{2}+i \sin\frac{3\pi}{2}\right)$$
$$= 2\left(0+i\cdot(-1)\right) = -2i\ .$$