Math 21C (Spring 2005)
Kouba
Exam 1

Please PRINT your name here : ANSWER KEY

Please SIGN your name here : ______________________

Your Exam ID Number ________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 9:50 a.m. sharp to finish the exam.

7. Make sure that you have 5 pages including the cover page..

1.) (10 pts.) Determine the *distance* from the point $(0, 2, -4)$ to the *center* of the sphere given by $x^2 - 4x + y^2 + 6y + z^2 = 100$.

$$(x^2-4x+4)+(y^2+6y+9)+z^2 = 100+4+9 \rightarrow$$
$$(x-2)^2+(y+3)^2+(z-0)^2 = 113 \rightarrow$$

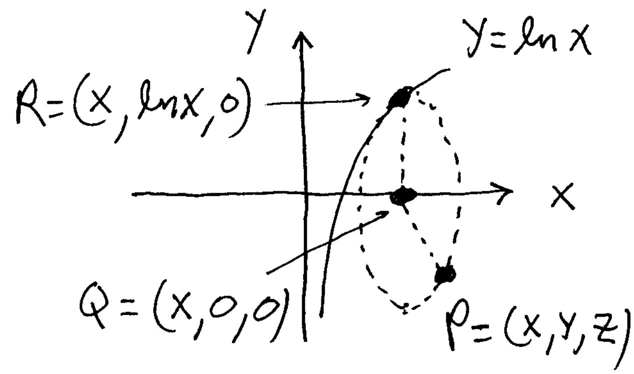
center : $(2,-3,0)$ so distance from $(0,2,-4)$ is

$$L = \sqrt{(2-0)^2+(-3-2)^2+(0--4)^2} = \sqrt{45}$$

2.) (10 pts.) Find an equation of the surface in three-dimensional space formed by revolving the graph of the equation $y = \ln x$ about the x-axis.

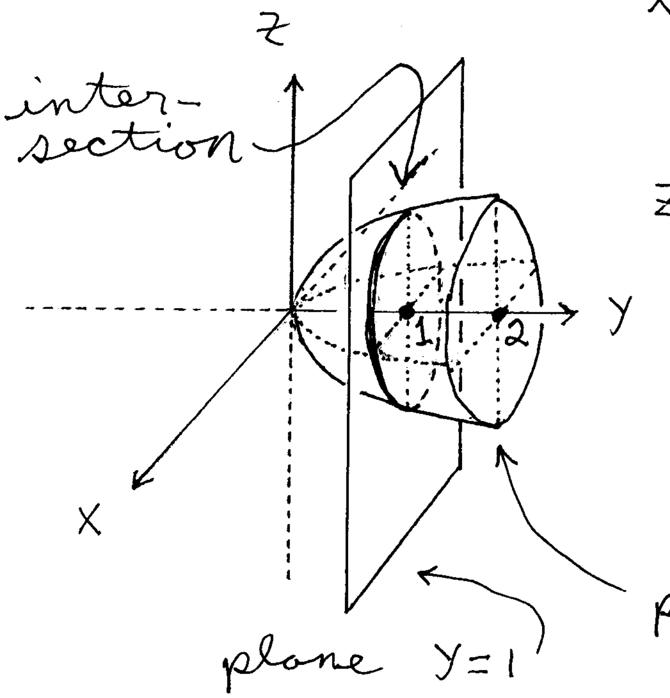
Set radii equal :

$$\sqrt{(x-x)^2+(y-0)^2+(z-0)^2}$$
$$= \sqrt{(x-x)^2+(\ln x-0)^2+(0-0)^2}$$

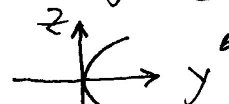


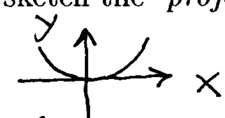
$$\rightarrow \quad y^2+z^2 = (\ln x)^2$$

3.) (12 pts.) Sketch the surfaces $y = x^2 + z^2$ and $y = 1$ and their *intersection* on the same set of axes in three-dimensional space. On another set of axes sketch the *projection* of this intersection in the xz-plane.



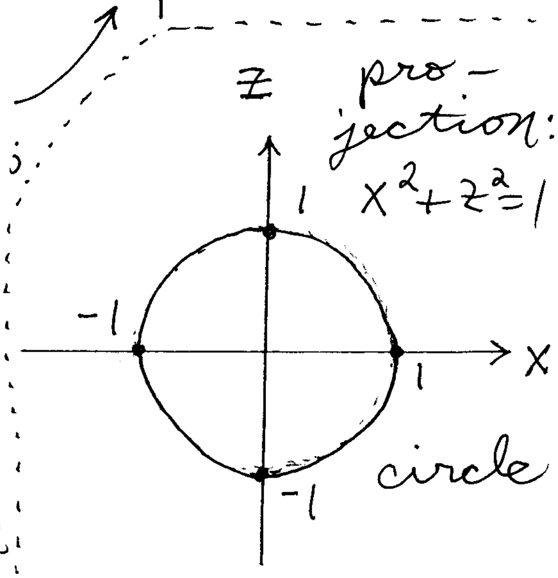
$x=0$: $y = z^2$;





$z=0$: $y = x^2$;

$y=0$:
$x^2+z^2=0$
$\rightarrow x=0,\ z=0$.



4.) Evaluate each of the following limits or determine that the limit does not exist.

a.) (10 pts.) $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{x^2 - y^4}{x + y^2}$ $\text{“}\frac{0}{0}\text{”}$ $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{(x-y^2)\cancel{(x+y^2)}}{\cancel{x+y^2}}$

$= (-1) - (1)^2 = -2$

b.) (10 pts.) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4 + y^4} = \text{“}\frac{0}{0}\text{”}$? Limit DNE since:

<u>path $x=0$</u> : $\displaystyle\lim_{(x,y)\to(0,0)} \frac{0}{y^4} = \lim_{(x,y)\to(0,0)} 0 = 0$

<u>path $y=x$</u> : $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4}{x^4+x^4} = \lim_{(x,y)\to(0,0)} \frac{\cancel{x^4}}{2\cancel{x^4}} = \frac{1}{2}$

5.) (10 pts.) Determine the domain and range for the following function and sketch (shade) the domain in the xy-plane : $f(x,y) = \sqrt{25 - x^2 - y^2}$.

$25 - x^2 - y^2 \geq 0 \rightarrow x^2 + y^2 \leq 5^2$

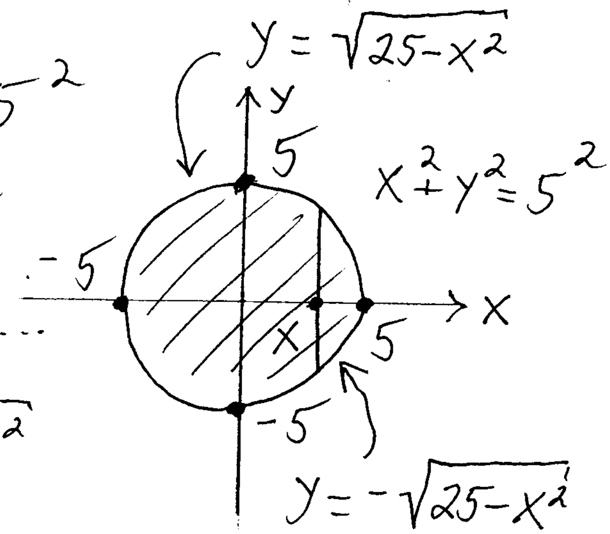
(all points on or inside the circle $x^2+y^2=5^2$)

$z = \sqrt{25-x^2-y^2} \geq 0 \rightarrow x^2+y^2+z^2 = 5^2$

sphere

Domain : $\begin{cases} -5 \leq x \leq 5 \\ -\sqrt{25-x^2} \leq y \leq \sqrt{25-x^2} \end{cases}$

Range : $0 \leq z \leq 5$



6.) (12 pts.) Show that $u(x,t) = e^{-t}\sin x$ satisfies the equation $u_{xx} = u_t$.

$u_x = e^{-t}\cos x$, $u_{xx} = e^{-t} \cdot -\sin x = -e^{-t}\sin x$,

$u_t = -e^{-t}\sin x$; then $u_{xx} = u_t$.

7.) (16 pts.) Find and classify (relative maximum, relative minimum, or saddle point) the critical points for $z = 2x^4 - x^2 + 3y^2$.

$z_x = 8x^3 - 2x = 2x(4x^2 - 1) = 0 \rightarrow \boxed{x=0}$ or $\boxed{x = \pm\frac{1}{2}}$;

$z_y = 6y = 0 \rightarrow \boxed{y=0}$; then critical points are $(0,0)$, $(\frac{1}{2},0)$, and $(\frac{-1}{2},0)$;

$$z_{xx} = 24x^2 - 2, \quad z_{yy} = 6, \quad z_{xy} = 0 \quad ;$$

check $(0,0)$: $D = z_{xx} z_{yy} - z_{xy}^2 = (-2)(6) - (0)^2 = -12 < 0$

so $(0,0)$ determines a $\boxed{\text{saddle point}}$ (with $z = 0$) ;

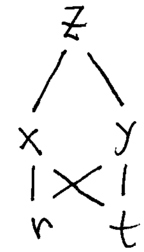
check $(\frac{1}{2},0)$: $D = z_{xx} z_{yy} - z_{xy}^2 = (4)(6) - (0)^2 = 24 > 0$

and $z_{xx} = 4 > 0$ so $(\frac{1}{2},0)$ determines a $\boxed{\text{minimum}}$ value at $z = -1/8$;

check $(\frac{-1}{2},0)$: $D = z_{xx} z_{yy} - z_{xy}^2 = (4)(6) - (0)^2 = 24 > 0$

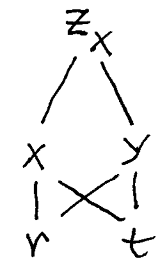
and $z_{xx} = 4 > 0$ so $(\frac{-1}{2},0)$ determines a $\boxed{\text{minimum}}$ value at $z = \frac{-1}{8}$.

8.) (10 pts.) Assume that $z = f(x, y)$, $x = rt$, and $y = 2t - r^2$. Compute the second partial derivative $\dfrac{\partial^2 z}{\partial r^2}$. SIMPLIFY your final answer.



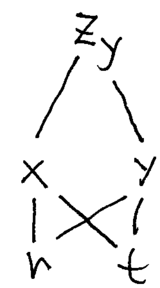
$$\frac{\partial z}{\partial r} = z_x \cdot \frac{\partial x}{\partial r} + z_y \cdot \frac{\partial y}{\partial r} = z_x \cdot (t) + z_y \cdot (-2r) \longrightarrow$$

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}\left[z_x \cdot (t) - z_y \cdot (2r)\right]$$



$$= \left\{ z_x \cdot \frac{\partial}{\partial r}(t) + \frac{\partial}{\partial r}(z_x) \cdot (t) \right\} - \left\{ z_y \cdot \frac{\partial}{\partial r}(2r) + \frac{\partial}{\partial r}(z_y) \cdot (2r) \right\}$$

$$= z_x \cdot (0) + \left[z_{xx} \cdot \frac{\partial x}{\partial r} + z_{xy} \cdot \frac{\partial y}{\partial r} \right] \cdot (t) - z_y \cdot (2) - \left[z_{yx} \cdot \frac{\partial x}{\partial r} + z_{yy} \cdot \frac{\partial y}{\partial r} \right] \cdot (2r)$$



$$= \left[z_{xx} \cdot (t) + z_{xy} \cdot (-2r) \right] \cdot (t) - 2 z_y - \left[z_{xy} \cdot (t) + z_{yy} \cdot (-2r) \right] \cdot (2r)$$

$$= z_{xx} \cdot (t^2) - z_{xy} \cdot (4rt) + z_{yy} \cdot (4r^2) - 2 z_y$$

The following EXTRA CREDIT PROBLEM is worth 12 points. This problem is OPTIONAL.

1.) Determine the value of $f_y(0,0)$ for the following function :

$$f(x,y) = \begin{cases} \dfrac{\sin(x + y^3)}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) . \end{cases}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h} \longrightarrow$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0, 0+h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(0,h) - 0}{h}$$

$$= \lim_{h \to 0} \frac{\sin(h^3)}{h^2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{\sin(h^3)}{h^3} = 1 .$$