Math 21C (Spring 2005)
Kouba
Exam 2

Please PRINT your name here : ______KEY______

Please SIGN your name here : ______________________

Your Exam ID Number ________

1. PLEASE DO NOT TURN THIS PAGE UNTIL TOLD TO DO SO.

2. IT IS A VIOLATION OF THE UNIVERSITY HONOR CODE TO, IN ANY WAY, ASSIST ANOTHER PERSON IN THE COMPLETION OF THIS EXAM. PLEASE KEEP YOUR OWN WORK COVERED UP AS MUCH AS POSSIBLE DURING THE EXAM SO THAT OTHERS WILL NOT BE TEMPTED OR DISTRACTED. THANK YOU FOR YOUR COOPERATION.

3. YOU MAY USE A CALCULATOR ON THIS EXAM.

4. No notes, books, or classmates may be used as resources for this exam.

5. Read directions to each problem carefully. Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your getting a good score on this exam. Neatness and organization are also important.

6. You have until 9:50 a.m. sharp to finish the exam. PLEASE STOP WRITING IMMEDIATELY when time is called and close your exam.

7. **Make sure that you have 5 pages including the cover page.**

1.) (5 pts. each) Consider the flat region R enclosed by the graphs of $y = 2x$, $x = 0$, and the line $y = 6$. Sketch the region and describe R using

a.) vertical cross sections.

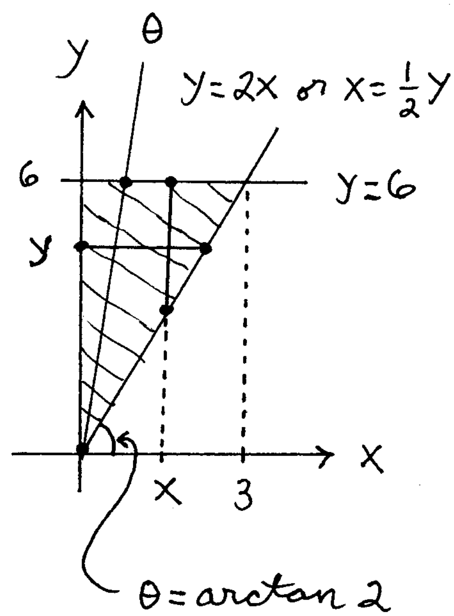
$$\begin{cases} 0 \le x \le 3 \\ 2x \le y \le 6 \end{cases}$$

b.) horizontal cross sections.

$$\begin{cases} 0 \le y \le 6 \\ 0 \le x \le \frac{1}{2} y \end{cases}$$

c.) polar coordinates.

$$\begin{cases} \arctan 2 \le \theta \le \pi/2 \\ 0 \le r \le 6 \csc\theta \end{cases}$$



$y = 6 \to r \sin\theta = 6$

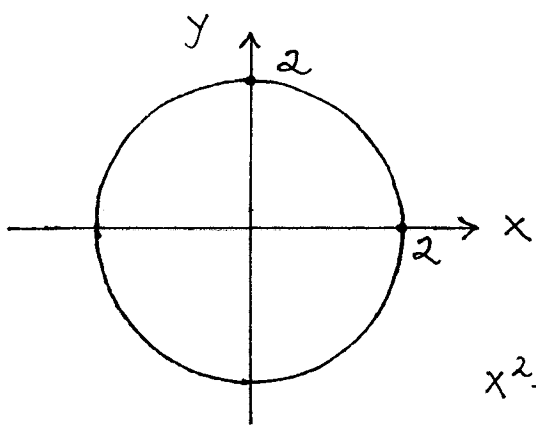
$\to r = 6 \csc\theta$

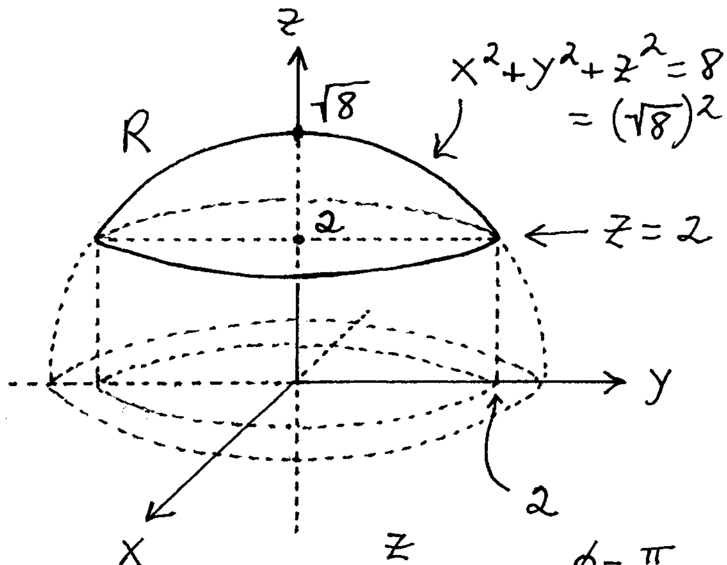
2.) (12 pts.) Consider the solid region R inside the sphere $x^2 + y^2 + z^2 = 8$ and above the plane $z = 2$. Sketch the region and describe R using spherical coordinates.

$x^2 + y^2 + z^2 = 8$ and $z = 2$

$\to x^2 + y^2 + 4 = 8 \to$

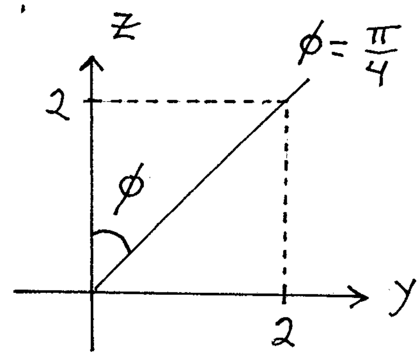
$\to x^2 + y^2 = 4 = 2^2$





$x^2 + y^2 + z^2 = 8$

$\to \rho^2 = 8 \to \rho = \sqrt{8}$



$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi/4 \\ 2 \sec\phi \le \rho \le \sqrt{8} \end{cases}$$

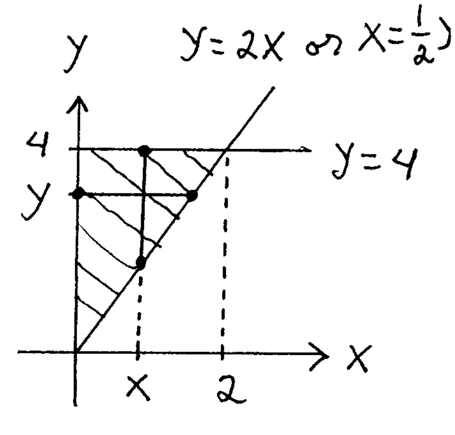
$z = 2$

$\to \rho \cos\phi = 2$

$\to \rho = 2 \sec\phi$

3.) (12 pts. each) Evaluate each of the following double integrals.

a.) $\displaystyle\int_0^2 \int_{2x}^4 e^{y^2}\, dy\, dx$ (HINT: Reverse the order of integration.)

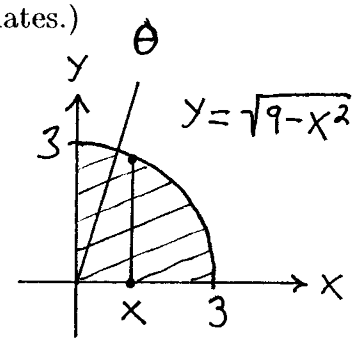


$$= \int_0^4 \int_0^{\frac{1}{2}y} e^{y^2}\, dx\, dy$$

$$= \int_0^4 x e^{y^2} \Big|_{x=0}^{x=\frac{1}{2}y}\, dy$$

$$= \int_0^4 \frac{1}{2} y e^{y^2}\, dy = \frac{1}{4} e^{y^2} \Big|_0^4 = \frac{1}{4}\left(e^{16} - 1\right)$$

b.) $\displaystyle\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{x^2+y^2}\, dy\, dx$ (HINT: Convert to polar coordinates.)



$$= \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 r^2\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1}{3} r^3 \Big|_0^3 \right) d\theta = \int_0^{\frac{\pi}{4}} 9\, d\theta$$

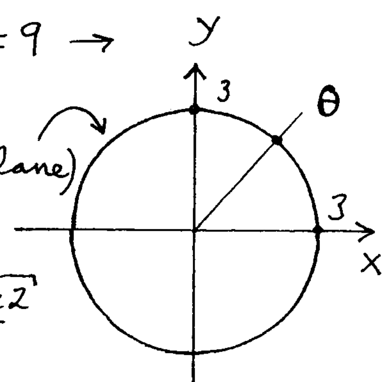
$$= 9\theta \Big|_0^{\frac{\pi}{2}} = \frac{9}{2}\pi$$

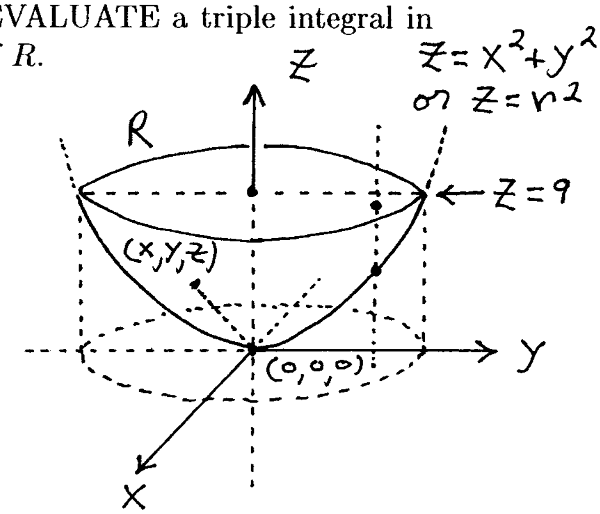
4.) (12 pts.) Consider the solid region R enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$. Assume that the density at point $P = (x, y, z)$ is numerically equal to the distance from P to the origin. SET UP BUT DO NOT EVALUATE a triple integral in cylindrical coordinates which represents the *total mass* of R.

$z = x^2 + y^2$ and $z = 9 \rightarrow$

$x^2 + y^2 = 9 = 3^2$

(projection on xy-plane)





density

$\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$

$$\text{Mass} = \int_R \delta(P)\, dV$$

$$= \int_0^{2\pi} \int_0^3 \int_{r^2}^9 \sqrt{r^2+z^2} \cdot r\, dz\, dr\, d\theta$$

5.) (13 pts.) Set up and EVALUATE a triple integral (using your choice of coordinate system) representing the *volume* of a right circular cylinder of radius a and height h.

Using cylindrical coordinates:

$$\text{Vol} = \int_R 1\, dV$$

$$= \int_0^{2\pi}\int_0^{a}\int_0^{h} 1\cdot r\, dz\, dr\, d\theta$$

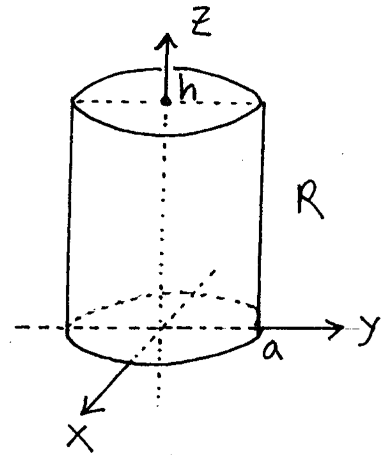
$$= \int_0^{2\pi}\int_0^{a} \left(rz \Big|_{z=0}^{z=h} \right) dr\, d\theta$$

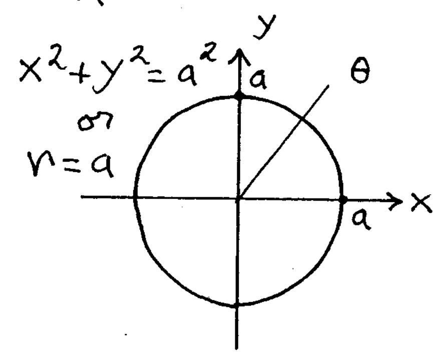
$$= \int_0^{2\pi}\int_0^{a} rh\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^2}{2} h \Big|_{r=0}^{r=a} \right) d\theta$$

$$= \int_0^{2\pi} \frac{a^2}{2} h\, d\theta$$

$$= \frac{a^2}{2} h\, \theta \Big|_{\theta=0}^{\theta=2\pi} = \pi a^2 h$$





6.) (12 pts.) Consider the solid region R enclosed by the paraboloid $z = 3 - x^2 - y^2$ and the plane $z = 2x$. Assume that the density at point $P = (x, y, z)$ is given by $\delta(x, y, z) = y^2 + z^2$. SET UP BUT DO NOT EVALUATE a triple integral in rectangular coordinates which represents the *moment of inertia* of R about the z-axis.

$z = 3 - x^2 - y^2$ and $z = 2x$

$\rightarrow\ 2x = 3 - x^2 - y^2$

$\rightarrow\ x^2 + 2x + 1 + y^2 = 3 + 1$

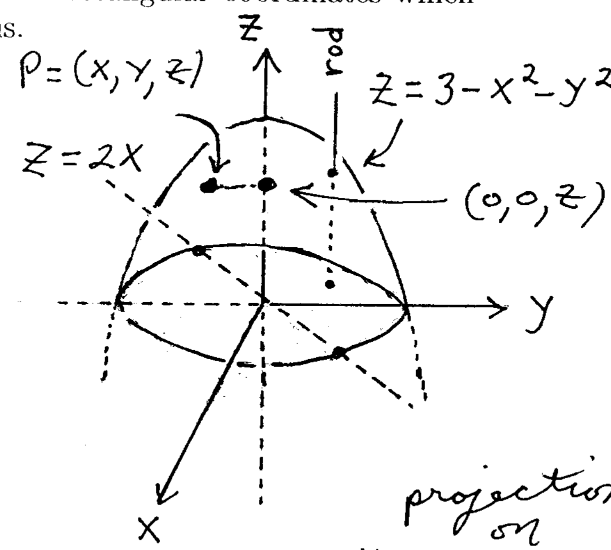
$\rightarrow\ (x+1)^2 + y^2 = 2^2$; then

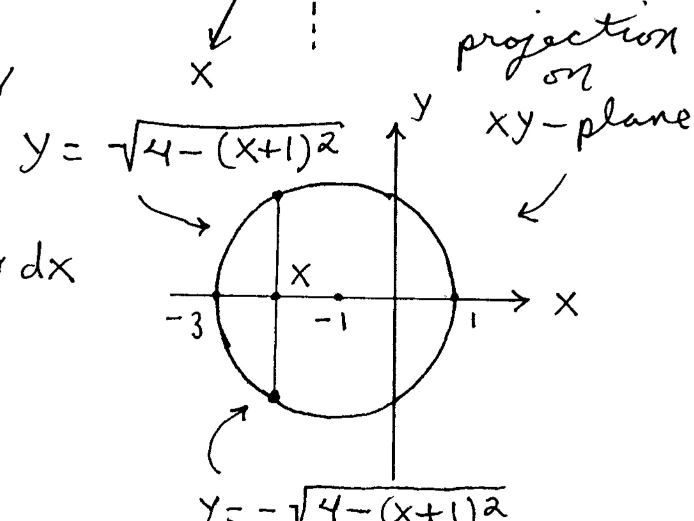
$$\text{distance} = \sqrt{(x-0)^2 + (y-0)^2 + (z-z)^2}$$

$$= \sqrt{x^2+y^2} \quad \text{and}$$

$$\text{M. of I.} = \int_R (\text{distance})^2 (\text{density})\, dV$$

$$= \int_{-3}^{1}\int_{-\sqrt{4-(x+1)^2}}^{\sqrt{4-(x+1)^2}}\int_{2x}^{3-x^2-y^2} \left(\sqrt{x^2+y^2}\right)^2 \cdot (y^2+z^2)\, dz\, dy\, dx$$



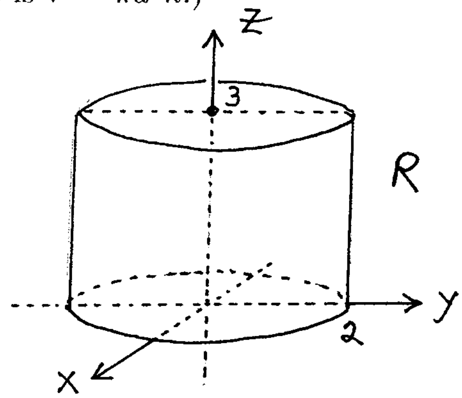


7.) (12 pts.) Consider the solid region R enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 3$. SET UP BUT DO NOT EVALUATE a triple integral in spherical coordinates which represents the *average value* of function $f(x, y, z) = y + z$ over region R. (HINT: The volume of a cylinder of radius a and height h is $V = \pi a^2 h$.)

$z = 3 \to \rho\cos\phi = 3 \to \rho = 3\sec\phi$;

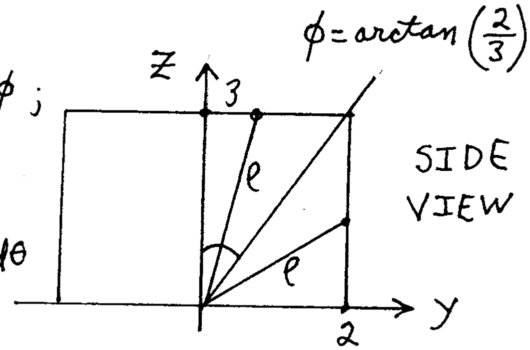
$x^2 + y^2 = 4 \to r^2 = 4 \to r = 2$

$\to \rho\sin\phi = 2 \to \rho = 2\csc\phi$;



$$R: \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \phi \le \arctan\left(\frac{2}{3}\right) \\ 0 \le \rho \le 3\sec\phi \end{cases} ; \quad \begin{cases} 0 \le \theta \le 2\pi \\ \arctan\left(\frac{2}{3}\right) \le \phi \le \frac{\pi}{2} \\ 0 \le \rho \le 2\csc\phi \end{cases}$$

Vol. of $R = \pi(2)^2(3)$; $f(x,y,z) = y + z = \rho\sin\theta\sin\phi + \rho\cos\phi$;



$$\text{AVE} = \frac{1}{\text{Vol.} R} \int_R f(P)\, dV$$

$$= \frac{1}{12\pi} \left\{ \int_0^{2\pi} \int_0^{\arctan(2/3)} \int_0^{3\sec\phi} (\rho\sin\theta\sin\phi + \rho\cos\phi) \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \right.$$

$$\left. + \int_0^{2\pi} \int_{\arctan(2/3)}^{\pi/2} \int_0^{2\csc\phi} (\rho\sin\theta\sin\phi + \rho\cos\phi) \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \right\}$$

The following EXTRA CREDIT PROBLEM is worth 12 points. This problem is OPTIONAL.

1.) Convert the following integral to rectangular coordinates. DO NOT EVALUATE THE INTEGRAL.

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\phi} \rho^3 \cos\theta \sin^2\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\cos\phi} \underbrace{(\rho\cos\theta\sin\phi)}_{x} \cdot \underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV}$$

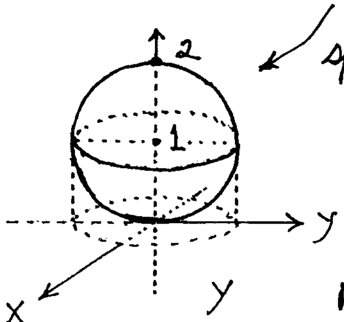
$\rho = 2\cos\phi \to \rho^2 = 2\rho\cos\phi$

$\to x^2 + y^2 + z^2 = 2z$

$\to x^2 + y^2 + (z-1)^2 = 1$;

sphere

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} x \; dz\, dy\, dx$$



projection on xy-plane