Math 21C
Kouba
Taylor Series and Taylor Polynomials

<u>QUESTION</u> : What connection do ordinary functions, $y = f(x)$, have to power series centered at $x = a$ of the form $\sum_{n=0}^{\infty} a_n(x-a)^n$?

<u>ANSWER</u> : Assume that $y = f(x)$ is a given function and constant "a" is known. Determine a sequence of real numbers $\{a_n\}$ so that

$$\text{(T)} \quad f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots .$$

If we substitute $x = a$ in equation (T), we get

$$f(a) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots = a_0 ,$$

i.e.,

$$a_0 = f(a) .$$

Now differentiate equation (T) term by term getting

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + \cdots .$$

If we substitute $x = a$ in this equation, we get

$$f'(a) = a_1 + 2a_2(0) + 3a_3(0)^2 + 4a_4(0)^3 + \cdots = a_1 ,$$

i.e.,

$$a_1 = f'(a) .$$

Now differentiate again term by term getting

$$f''(x) = 2a_2 + 3 \cdot 2a_3(x-a) + 4 \cdot 3a_4(x-a)^2 + 5 \cdot 4a_5(x-a)^3 + \cdots .$$

If we substitute $x = a$ in this equation, we get

$$f''(a) = 2a_2 + 3 \cdot 2a_3(0) + 4 \cdot 3a_4(0)^2 + 5 \cdot 4a_4(0)^3 + \cdots = 2a_2 ,$$

i.e.,

$$a_2 = \frac{f''(a)}{2!} .$$

Now differentiate again term by term getting

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x-a) + 5 \cdot 4 \cdot 3a_5(x-a)^2 + 6 \cdot 5 \cdot 4a_6(x-a)^3 + \cdots .$$

If we substitute $x = a$ in this equation, we get

$$f'''(a) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(0) + 5 \cdot 4 \cdot 3a_5(0)^2 + 6 \cdot 5 \cdot 4a_6(0)^3 + \cdots = 3 \cdot 2a_3 ,$$

i.e.,

$$a_3 = \frac{f'''(a)}{3!} .$$

Continuing this term by term differentiation and substitution process results in the fact that

$$\text{(S)} \qquad a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n = 0, 1, 2, 3, 4, \cdots .$$

DEFINITION : Equations (T) and (S) together are called the *Taylor Series* for function $y = f(x)$ centered at $x = a$. For the special case of $a = 0$, we call the series a *Maclaurin Series* .

DEFINITION : The *Taylor Polynomial of degree* n centered at $x = a$ for function $y = f(x)$ is given by

$$\text{(P)} \quad P_n(x; a) = \sum_{k=0}^{n} a_k (x-a)^k = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n$$

and

$$\text{(S)} \qquad a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for } k = 0, 1, 2, 3, \cdots, n .$$

REMARK : The *Taylor Polynomial of degree* n centered at $x = a$ for function $y = f(x)$ is the *Taylor Series* centered at $x = a$ terminated at the nth power of $(x - a)$. It is NOT defined to be the first n or $n + 1$ terms of the Taylor Series.

QUESTION : For what x-values is an ordinary function $y = f(x)$ equal to its Taylor series centered at $x = a$, i.e, for what x-values is $y = f(x)$ equal to

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots \quad ?$$

ANSWER : Let $P_n(x; a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ be the *Taylor Polynomial of degree* n centered at $x = a$ and let $R_n(x; a) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \cdots$, which is simply the remaining infinite tail of the Taylor Series centered at $x = a$. It can be shown that

$$R_n(x; a) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-a)^{(n+1)} ,$$

where c_n is between numbers a and x. This is called the Lagrange form of the Taylor remainder. Those x-values for which $y = f(x)$ is equal to its Taylor series are precisely those x-values for which

$$\lim_{n \to \infty} R_n(x; a) = 0 .$$