Math 21C
Kouba

Testing Infinite Series for Convergence/Divergence

1.) nth term test (for divergence only): If $\lim_{n \to \infty} a_n \neq 0$, then the series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots \quad \text{diverges}.$$

2.) geometric series test: The series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots + r^n + \cdots = \frac{1}{1-r} \quad \text{for } -1 < r < 1.$$

This series diverges for all other values of r. Note also that $1 + r + r^2 + \cdots + r^n = \dfrac{1 - r^{n+1}}{1-r}$ for $r \neq 1$.

3.) p-series test: The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

a.) converges if $p > 1$.
b.) diverges if $p \le 1$.

4.) integral test: assume function f is cont., positive, and decreasing for $x \ge 1$, and consider the series $\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \cdots$.

a.) If $\int_1^{\infty} f(x)\,dx$ converges, then the series converges.

b.) If $\int_1^{\infty} f(x)\,dx$ diverges, then the series diverges.

Let $R_n = f(n+1) + f(n+2) + f(n+3) + \cdots$. Then

$$\int_{n+1}^{\infty} f(x)\,dx < R_n < \int_{n}^{\infty} f(x)\,dx .$$

5.) sequence of partial sums: Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + a_{n+1} + \cdots$ and let

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$ be a partial sum.

a.) If $\lim_{n\to\infty} S_n = L$, then $\sum_{n=1}^{\infty} a_n = L$.

b.) If $\lim_{n\to\infty} S_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

6.) comparison test: Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$.

a.) If $0 \le a_n \le c_n$ and $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

b.) If $0 \le d_n \le a_n$ and $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

7.) limit comparison test: Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ with $a_n \ge 0$.

a.) If $\sum_{n=1}^{\infty} c_n$ converges with $c_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{c_n} = L$ (a nonzero number), then $\sum_{n=1}^{\infty} a_n$ converges. If $L = 0$, then $\sum_{n=1}^{\infty} a_n$ converges. If $L = +\infty$, then no conclusion can be made.

b.) If $\sum_{n=1}^{\infty} d_n$ diverges with $d_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{d_n} = L$ (a nonzero number), then $\sum_{n=1}^{\infty} a_n$ diverges. If $L = +\infty$, then $\sum_{n=1}^{\infty} a_n$ diverges. If $L = 0$, then no conclusion can be made.

8.) <u>alternating series test</u>: Assume that a_n is positive and decreasing with $\lim_{n\to\infty} a_n = 0$. Then the series

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 - \cdots + (-1)^n a_n + \cdots$$

converges. Let

$$R_n = (-1)^{n+1} a_{n+1} + (-1)^{n+2} a_{n+2} + \cdots .$$

Then $|R_n| < a_{n+1}$.

9.) <u>absolute convergence test</u> : Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$,

which may include negative terms.
If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

10.) <u>absolute ratio test</u> : Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots$ and let

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L .$$

a.) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
b.) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
c.) If $L = 1$, then no conclusion can be made.

11.) <u>absolute root test</u> : Consider the series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ and let

$$\lim_{n \to \infty} \left(|a_n|\right)^{\frac{1}{n}} = L .$$

a.) If $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
b.) If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
c.) If $L = 1$, then no conclusion can be made.