

Math 21C
 Kouba
 Absolute Convergence Test

Absolute Convergence Test : Consider the series $\sum_{n=1}^{\infty} a_n$, which has both positive and negative terms. If $\sum_{n=1}^{\infty} |a_n|$ converges ($< \infty$), then $\sum_{n=1}^{\infty} a_n$ converges.

Proof : Consider that for $n = 1, 2, 3, 4, \dots$

$$a_n + |a_n| = \begin{cases} 2 \cdot |a_n|, & \text{if } a_n > 0 \\ 0, & \text{if } a_n < 0. \end{cases}$$

Thus,

$$0 \leq a_n + |a_n| \leq 2 \cdot |a_n|$$

for $n = 1, 2, 3, 4, \dots$. But the series $\sum_{n=1}^{\infty} 2 \cdot |a_n| = 2 \cdot \sum_{n=1}^{\infty} |a_n|$ converges (Since scalar multiples of convergent series are convergent.), so that $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges by the

Comparison Test. We also have that $\sum_{n=1}^{\infty} -|a_n|$ converges (Since scalar multiples of convergent series are convergent.). It follows that

$$\sum_{n=1}^{\infty} (a_n + |a_n|) + \sum_{n=1}^{\infty} (-|a_n|) = \sum_{n=1}^{\infty} a_n$$

converges since the sum of convergent series is convergent. This completes the proof.