It can be shown that the series also converges to $\tan^{-1} x$ at the endpoints $x = \pm 1$, but we omit the proof. ∎

Notice that the original series in Example 5 converges at both endpoints of the original interval of convergence, but Theorem 22 can guarantee the convergence of the differentiated series only inside the interval.

EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$. Therefore,

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x \qquad \text{Theorem 22}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

or

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

It can also be shown that the series converges at $x = 1$ to the number $\ln 2$, but that was not guaranteed by the theorem. ∎

Alternating Harmonic Series Sum

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Exercises 10.7

Intervals of Convergence

In Exercises 1–36, **(a)** find the series' radius and interval of convergence. For what values of x does the series converge **(b)** absolutely, **(c)** conditionally?

1. $\sum_{n=0}^{\infty} x^n$
2. $\sum_{n=0}^{\infty} (x+5)^n$
3. $\sum_{n=0}^{\infty} (-1)^n (4x+1)^n$
4. $\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n}$
5. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{10^n}$
6. $\sum_{n=0}^{\infty} (2x)^n$
7. $\sum_{n=0}^{\infty} \frac{nx^n}{n+2}$
8. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$
9. $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\,3^n}$
10. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$
11. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$
12. $\sum_{n=0}^{\infty} \frac{3^n x^n}{n!}$
13. $\sum_{n=1}^{\infty} \frac{4^n x^{2n}}{n}$
14. $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$
15. $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n^2+3}}$
16. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n+3}}$
17. $\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$
18. $\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$
19. $\sum_{n=0}^{\infty} \frac{\sqrt{n}x^n}{3^n}$
20. $\sum_{n=1}^{\infty} \sqrt[n]{n}(2x+5)^n$
21. $\sum_{n=1}^{\infty} (2 + (-1)^n) \cdot (x+1)^{n-1}$
22. $\sum_{n=1}^{\infty} \frac{(-1)^n 3^{2n} (x-2)^n}{3n}$
23. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n$
24. $\sum_{n=1}^{\infty} (\ln n) x^n$
25. $\sum_{n=1}^{\infty} n^n x^n$
26. $\sum_{n=0}^{\infty} n!(x-4)^n$
27. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n2^n}$
28. $\sum_{n=0}^{\infty} (-2)^n (n+1)(x-1)^n$
29. $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$ Get the information you need about $\sum 1/(n(\ln n)^2)$ from Section 10.3, Exercise 55.
30. $\sum_{n=2}^{\infty} \frac{x^n}{n \ln n}$ Get the information you need about $\sum 1/(n \ln n)$ from Section 10.3, Exercise 54.
31. $\sum_{n=1}^{\infty} \frac{(4x-5)^{2n+1}}{n^{3/2}}$
32. $\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$

33. $\sum_{n=1}^{\infty} \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n)} x^n$

34. $\sum_{n=1}^{\infty} \frac{3 \cdot 5 \cdot 7 \cdots (2n+1)}{n^2 \cdot 2^n} x^{n+1}$

35. $\sum_{n=1}^{\infty} \frac{1 + 2 + 3 + \cdots + n}{1^2 + 2^2 + 3^2 + \cdots + n^2} x^n$

36. $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})(x-3)^n$

In Exercises 37–40, find the series' radius of convergence.

37. $\sum_{n=1}^{\infty} \frac{n!}{3 \cdot 6 \cdot 9 \cdots 3n} x^n$

38. $\sum_{n=1}^{\infty} \left(\frac{2 \cdot 4 \cdot 6 \cdots (2n)}{2 \cdot 5 \cdot 8 \cdots (3n-1)} \right)^2 x^n$

39. $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^n (2n)!} x^n$

40. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2} x^n$

(*Hint:* Apply the Root Test.)

In Exercises 41–48, use Theorem 20 to find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

41. $\sum_{n=0}^{\infty} 3^n x^n$

42. $\sum_{n=0}^{\infty} (e^x - 4)^n$

43. $\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{4^n}$

44. $\sum_{n=0}^{\infty} \frac{(x+1)^{2n}}{9^n}$

45. $\sum_{n=0}^{\infty} \left(\frac{\sqrt{x}}{2} - 1 \right)^n$

46. $\sum_{n=0}^{\infty} (\ln x)^n$

47. $\sum_{n=0}^{\infty} \left(\frac{x^2+1}{3} \right)^n$

48. $\sum_{n=0}^{\infty} \left(\frac{x^2-1}{2} \right)^n$

Using the Geometric Series

49. In Example 2 we represented the function $f(x) = 2/x$ as a power series about $x = 2$. Use a geometric series to represent $f(x)$ as a power series about $x = 1$, and find its interval of convergence.

50. Use a geometric series to represent each of the given functions as a power series about $x = 0$, and find their intervals of convergence.

   a. $f(x) = \frac{5}{3-x}$ **b.** $g(x) = \frac{3}{x-2}$

51. Represent the function $g(x)$ in Exercise 50 as a power series about $x = 5$, and find the interval of convergence.

52. **a.** Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{8}{4^{n+2}} x^n.$$

   b. Represent the power series in part (a) as a power series about $x = 3$ and identify the interval of convergence of the new series. (Later in the chapter you will understand why the new interval of convergence does not necessarily include all of the numbers in the original interval of convergence.)

Theory and Examples

53. For what values of x does the series

$$1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converge? What is its sum? What series do you get if you differentiate the given series term by term? For what values of x does the new series converge? What is its sum?

54. If you integrate the series in Exercise 53 term by term, what new series do you get? For what values of x does the new series converge, and what is another name for its sum?

55. The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

converges to $\sin x$ for all x.

   a. Find the first six terms of a series for $\cos x$. For what values of x should the series converge?

   b. By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.

   c. Using the result in part (a) and series multiplication, calculate the first six terms of a series for $2 \sin x \cos x$. Compare your answer with the answer in part (b).

56. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

converges to e^x for all x.

   a. Find a series for $(d/dx)e^x$. Do you get the series for e^x? Explain your answer.

   b. Find a series for $\int e^x\,dx$. Do you get the series for e^x? Explain your answer.

   c. Replace x by $-x$ in the series for e^x to find a series that converges to e^{-x} for all x. Then multiply the series for e^x and e^{-x} to find the first six terms of a series for $e^{-x} \cdot e^x$.

57. The series

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots$$

converges to $\tan x$ for $-\pi/2 < x < \pi/2$.

   a. Find the first five terms of the series for $\ln|\sec x|$. For what values of x should the series converge?

   b. Find the first five terms of the series for $\sec^2 x$. For what values of x should this series converge?

   c. Check your result in part (b) by squaring the series given for $\sec x$ in Exercise 58.

58. The series

$$\sec x = 1 + \frac{x^2}{2} + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \cdots$$

converges to $\sec x$ for $-\pi/2 < x < \pi/2$.

   a. Find the first five terms of a power series for the function $\ln|\sec x + \tan x|$. For what values of x should the series converge?

b. Find the first four terms of a series for $\sec x \tan x$. For what values of x should the series converge?

c. Check your result in part (b) by multiplying the series for $\sec x$ by the series given for $\tan x$ in Exercise 57.

59. Uniqueness of convergent power series

a. Show that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ are convergent and equal for all values of x in an open interval $(-c, c)$, then $a_n = b_n$ for every n. (*Hint:* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$. Differentiate term by term to show that a_n and b_n both equal $f^{(n)}(0)/(n!)$.)

b. Show that if $\sum_{n=0}^{\infty} a_n x^n = 0$ for all x in an open interval $(-c, c)$, then $a_n = 0$ for every n.

60. The sum of the series $\sum_{n=0}^{\infty}(n^2/2^n)$ To find the sum of this series, express $1/(1 - x)$ as a geometric series, differentiate both sides of the resulting equation with respect to x, multiply both sides of the result by x, differentiate again, multiply by x again, and set x equal to $1/2$. What do you get?

10.8 Taylor and Maclaurin Series

We have seen how geometric series can be used to generate a power series for a few functions having a special form, like $f(x) = 1/(1 - x)$ or $g(x) = 3/(x - 2)$. Now we expand our capability to represent a function with a power series. This section shows how functions that are infinitely differentiable generate power series called *Taylor series*. In many cases, these series provide useful polynomial approximations of the generating functions. Because they are used routinely by mathematicians and scientists, Taylor series are considered one of the most important themes of infinite series.

Series Representations

We know from Theorem 21 that within its interval of convergence I the sum of a power series is a continuous function with derivatives of all orders. But what about the other way around? If a function $f(x)$ has derivatives of all orders on an interval, can it be expressed as a power series on at least part of that interval? And if it can, what are its coefficients?

We can answer the last question readily if we assume that $f(x)$ is the sum of a power series about $x = a$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

with a positive radius of convergence. By repeated term-by-term differentiation within the interval of convergence I, we obtain

$$\begin{aligned} f'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + \cdots + na_n(x-a)^{n-1} + \cdots, \\ f''(x) &= 1\cdot 2a_2 + 2\cdot 3a_3(x-a) + 3\cdot 4a_4(x-a)^2 + \cdots, \\ f'''(x) &= 1\cdot 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4(x-a) + 3\cdot 4\cdot 5a_5(x-a)^2 + \cdots, \end{aligned}$$

with the nth derivative, for all n, being

$$f^{(n)}(x) = n!a_n + \text{a sum of terms with } (x-a) \text{ as a factor.}$$

Since these equations all hold at $x = a$, we have

$$f'(a) = a_1, \qquad f''(a) = 1\cdot 2a_2, \qquad f'''(a) = 1\cdot 2\cdot 3a_3,$$

and, in general,

$$f^{(n)}(a) = n!a_n.$$

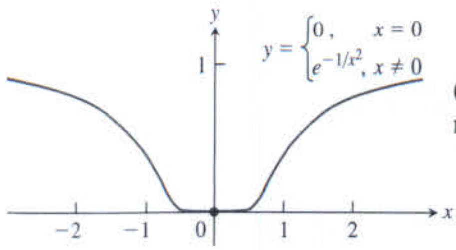

FIGURE 10.19 The graph of the continuous extension of $y = e^{-1/x^2}$ is so flat at the origin that all of its derivatives there are zero (Example 4). Therefore its Taylor series, which is zero everywhere, is not the function itself.

EXAMPLE 4 It can be shown (though not easily) that

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

(Figure 10.19) has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. This means that the Taylor series generated by f at $x = 0$ is

$$\begin{aligned} f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ = 0 + 0\cdot x + 0\cdot x^2 + \cdots + 0\cdot x^n + \cdots \\ = 0 + 0 + \cdots + 0 + \cdots. \end{aligned}$$

The series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$. That is, the Taylor series generated by $f(x)$ in this example is *not* equal to the function $f(x)$ over the entire interval of convergence. ■

Two questions still remain.

1. For what values of x can we normally expect a Taylor series to converge to its generating function?
2. How accurately do a function's Taylor polynomials approximate the function on a given interval?

The answers are provided by a theorem of Taylor in the next section.

Exercises 10.8

Finding Taylor Polynomials

In Exercises 1–10, find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a.

1. $f(x) = e^{2x}, \quad a = 0$

2. $f(x) = \sin x, \quad a = 0$

3. $f(x) = \ln x, \quad a = 1$

4. $f(x) = \ln(1 + x), \quad a = 0$

5. $f(x) = 1/x, \quad a = 2$

6. $f(x) = 1/(x + 2), \quad a = 0$

7. $f(x) = \sin x, \quad a = \pi/4$

8. $f(x) = \tan x, \quad a = \pi/4$

9. $f(x) = \sqrt{x}, \quad a = 4$

10. $f(x) = \sqrt{1 - x}, \quad a = 0$

Finding Taylor Series at $x = 0$ (Maclaurin Series)

Find the Maclaurin series for the functions in Exercises 11–22.

11. e^{-x}

12. xe^x

13. $\dfrac{1}{1 + x}$

14. $\dfrac{2 + x}{1 - x}$

15. $\sin 3x$

16. $\sin \dfrac{x}{2}$

17. $7\cos(-x)$

18. $5\cos \pi x$

19. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

20. $\sinh x = \dfrac{e^x - e^{-x}}{2}$

21. $x^4 - 2x^3 - 5x + 4$

22. $\dfrac{x^2}{x + 1}$

Finding Taylor and Maclaurin Series

In Exercises 23–32, find the Taylor series generated by f at $x = a$.

23. $f(x) = x^3 - 2x + 4, \quad a = 2$

24. $f(x) = 2x^3 + x^2 + 3x - 8, \quad a = 1$

25. $f(x) = x^4 + x^2 + 1, \quad a = -2$

26. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \quad a = -1$

27. $f(x) = 1/x^2, \quad a = 1$

28. $f(x) = 1/(1 - x)^3, \quad a = 0$

29. $f(x) = e^x, \quad a = 2$

30. $f(x) = 2^x, \quad a = 1$

31. $f(x) = \cos(2x + (\pi/2)), \quad a = \pi/4$

32. $f(x) = \sqrt{x + 1}, \quad a = 0$

In Exercises 33–36, find the first three nonzero terms of the Maclaurin series for each function and the values of x for which the series converges absolutely.

33. $f(x) = \cos x - (2/(1 - x))$

34. $f(x) = (1 - x + x^2)e^x$

35. $f(x) = (\sin x)\ln(1 + x)$

36. $f(x) = x\sin^2 x$

Theory and Examples

37. Use the Taylor series generated by e^x at $x = a$ to show that

$$e^x = e^a\left[1 + (x - a) + \frac{(x - a)^2}{2!} + \cdots\right].$$

38. (*Continuation of Exercise 37.*) Find the Taylor series generated by e^x at $x = 1$. Compare your answer with the formula in Exercise 37.

39. Let $f(x)$ have derivatives through order n at $x = a$. Show that the Taylor polynomial of order n and its first n derivatives have the same values that f and its first n derivatives have at $x = a$.

40. **Approximation properties of Taylor polynomials** Suppose that $f(x)$ is differentiable on an interval centered at $x = a$ and that $g(x) = b_0 + b_1(x - a) + \cdots + b_n(x - a)^n$ is a polynomial of degree n with constant coefficients $b_0, \ldots, b_n$. Let $E(x) = f(x) - g(x)$. Show that if we impose on g the conditions

i) $E(a) = 0$ — The approximation error is zero at $x = a$.

ii) $\lim_{x \to a} \frac{E(x)}{(x - a)^n} = 0,$ — The error is negligible when compared to $(x - a)^n$.

then

$$g(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Thus, the Taylor polynomial $P_n(x)$ is the only polynomial of degree less than or equal to n whose error is both zero at $x = a$ and negligible when compared with $(x - a)^n$.

Quadratic Approximations The Taylor polynomial of order 2 generated by a twice-differentiable function $f(x)$ at $x = a$ is called the *quadratic approximation* of f at $x = a$. In Exercises 41–46, find the **(a)** linearization (Taylor polynomial of order 1) and **(b)** quadratic approximation of f at $x = 0$.

41. $f(x) = \ln(\cos x)$

42. $f(x) = e^{\sin x}$

43. $f(x) = 1/\sqrt{1 - x^2}$

44. $f(x) = \cosh x$

45. $f(x) = \sin x$

46. $f(x) = \tan x$

10.9 Convergence of Taylor Series

In the last section we asked when a Taylor series for a function can be expected to converge to that (generating) function. We answer the question in this section with the following theorem.

> **THEOREM 23—Taylor's Theorem** If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that
>
> $$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{n+1}.$$

Taylor's Theorem is a generalization of the Mean Value Theorem (Exercise 45). There is a proof of Taylor's Theorem at the end of this section.

When we apply Taylor's Theorem, we usually want to hold a fixed and treat b as an independent variable. Taylor's formula is easier to use in circumstances like these if we change b to x. Here is a version of the theorem with this change.

> **Taylor's Formula**
>
> If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,
>
> $$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n(x), \tag{1}$$
>
> where
>
> $$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \tag{2}$$

Equations (6) and (9) give

$$f(b) = P_n(b) + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

This concludes the proof. ■

Exercises 10.9

Finding Taylor Series

Use substitution (as in Example 4) to find the Taylor series at $x = 0$ of the functions in Exercises 1–10.

1. e^{-5x} **2.** $e^{-x/2}$ **3.** $5\sin(-x)$

4. $\sin\left(\frac{\pi x}{2}\right)$ **5.** $\cos 5x^2$ **6.** $\cos\left(x^{2/3}/\sqrt{2}\right)$

7. $\ln(1+x^2)$ **8.** $\tan^{-1}(3x^4)$ **9.** $\dfrac{1}{1+\frac{3}{4}x^3}$

10. $\dfrac{1}{2-x}$

Use power series operations to find the Taylor series at $x = 0$ for the functions in Exercises 11–28.

11. xe^x **12.** $x^2\sin x$ **13.** $\dfrac{x^2}{2} - 1 + \cos x$

14. $\sin x - x + \dfrac{x^3}{3!}$ **15.** $x\cos \pi x$ **16.** $x^2\cos(x^2)$

17. $\cos^2 x$ (*Hint:* $\cos^2 x = (1 + \cos 2x)/2$.)

18. $\sin^2 x$ **19.** $\dfrac{x^2}{1-2x}$ **20.** $x\ln(1+2x)$

21. $\dfrac{1}{(1-x)^2}$ **22.** $\dfrac{2}{(1-x)^3}$ **23.** $x\tan^{-1}x^2$

24. $\sin x \cdot \cos x$ **25.** $e^x + \dfrac{1}{1+x}$ **26.** $\cos x - \sin x$

27. $\dfrac{x}{3}\ln(1+x^2)$ **28.** $\ln(1+x) - \ln(1-x)$

Find the first four nonzero terms in the Maclaurin series for the functions in Exercises 29–34.

29. $e^x\sin x$ **30.** $\dfrac{\ln(1+x)}{1-x}$ **31.** $(\tan^{-1}x)^2$

32. $\cos^2 x \cdot \sin x$ **33.** $e^{\sin x}$ **34.** $\sin(\tan^{-1}x)$

Error Estimates

35. Estimate the error if $P_3(x) = x - (x^3/6)$ is used to estimate the value of $\sin x$ at $x = 0.1$.

36. Estimate the error if $P_4(x) = 1 + x + (x^2/2) + (x^3/6) + (x^4/24)$ is used to estimate the value of e^x at $x = 1/2$.

37. For approximately what values of x can you replace $\sin x$ by $x - (x^3/6)$ with an error of magnitude no greater than 5×10^{-4}? Give reasons for your answer.

38. If $\cos x$ is replaced by $1 - (x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1 - (x^2/2)$ tend to be too large, or too small? Give reasons for your answer.

39. How close is the approximation $\sin x = x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin x$?

40. The estimate $\sqrt{1+x} = 1 + (x/2)$ is used when x is small. Estimate the error when $|x| < 0.01$.

41. The approximation $e^x = 1 + x + (x^2/2)$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

42. (*Continuation of Exercise 41.*) When $x < 0$, the series for e^x is an alternating series. Use the Alternating Series Estimation Theorem to estimate the error that results from replacing e^x by $1 + x + (x^2/2)$ when $-0.1 < x < 0$. Compare your estimate with the one you obtained in Exercise 41.

Theory and Examples

43. Use the identity $\sin^2 x = (1 - \cos 2x)/2$ to obtain the Maclaurin series for $\sin^2 x$. Then differentiate this series to obtain the Maclaurin series for $2\sin x\cos x$. Check that this is the series for $\sin 2x$.

44. (*Continuation of Exercise 43.*) Use the identity $\cos^2 x = \cos 2x + \sin^2 x$ to obtain a power series for $\cos^2 x$.

45. Taylor's Theorem and the Mean Value Theorem Explain how the Mean Value Theorem (Section 4.2, Theorem 4) is a special case of Taylor's Theorem.

46. Linearizations at inflection points Show that if the graph of a twice-differentiable function $f(x)$ has an inflection point at $x = a$, then the linearization of f at $x = a$ is also the quadratic approximation of f at $x = a$. This explains why tangent lines fit so well at inflection points.

47. The (second) second derivative test Use the equation

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(c_2)}{2}(x-a)^2$$

to establish the following test.

Let f have continuous first and second derivatives and suppose that $f'(a) = 0$. Then

a. f has a local maximum at a if $f'' \le 0$ throughout an interval whose interior contains a;

b. f has a local minimum at a if $f'' \ge 0$ throughout an interval whose interior contains a.

48. A cubic approximation Use Taylor's formula with $a = 0$ and $n = 3$ to find the standard cubic approximation of $f(x) = 1/(1 - x)$ at $x = 0$. Give an upper bound for the magnitude of the error in the approximation when $|x| \le 0.1$.

49. a. Use Taylor's formula with $n = 2$ to find the quadratic approximation of $f(x) = (1 + x)^k$ at $x = 0$ (k a constant).

b. If $k = 3$, for approximately what values of x in the interval $[0, 1]$ will the error in the quadratic approximation be less than $1/100$?

50. Improving approximations of π

a. Let P be an approximation of π accurate to n decimals. Show that $P + \sin P$ gives an approximation correct to $3n$ decimals. (*Hint:* Let $P = \pi + x$.)

T **b.** Try it with a calculator.

51. The Taylor series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is $\sum_{n=0}^{\infty} a_n x^n$ A function defined by a power series $\sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$ has a Taylor series that converges to the function at every point of $(-R, R)$. Show this by showing that the Taylor series generated by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is the series $\sum_{n=0}^{\infty} a_n x^n$ itself.

An immediate consequence of this is that series like

$$x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \cdots$$

and

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots,$$

obtained by multiplying Taylor series by powers of x, as well as series obtained by integration and differentiation of convergent power series, are themselves the Taylor series generated by the functions they represent.

52. Taylor series for even functions and odd functions (*Continuation of Section 10.7, Exercise 59.*) Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for all x in an open interval $(-R, R)$. Show that

a. If f is even, then $a_1 = a_3 = a_5 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only even powers of x.

b. If f is odd, then $a_0 = a_2 = a_4 = \cdots = 0$, i.e., the Taylor series for f at $x = 0$ contains only odd powers of x.

COMPUTER EXPLORATIONS

Taylor's formula with $n = 1$ and $a = 0$ gives the linearization of a function at $x = 0$. With $n = 2$ and $n = 3$ we obtain the standard quadratic and cubic approximations. In these exercises we explore the errors associated with these approximations. We seek answers to two questions:

a. For what values of x can the function be replaced by each approximation with an error less than 10^{-2}?

b. What is the maximum error we could expect if we replace the function by each approximation over the specified interval?

Using a CAS, perform the following steps to aid in answering questions (a) and (b) for the functions and intervals in Exercises 53–58.

Step 1: Plot the function over the specified interval.

Step 2: Find the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_3(x)$ at $x = 0$.

Step 3: Calculate the $(n + 1)$st derivative $f^{(n+1)}(c)$ associated with the remainder term for each Taylor polynomial. Plot the derivative as a function of c over the specified interval and estimate its maximum absolute value, M.

Step 4: Calculate the remainder $R_n(x)$ for each polynomial. Using the estimate M from Step 3 in place of $f^{(n+1)}(c)$, plot $R_n(x)$ over the specified interval. Then estimate the values of x that answer question (a).

Step 5: Compare your estimated error with the actual error $E_n(x) = |f(x) - P_n(x)|$ by plotting $E_n(x)$ over the specified interval. This will help answer question (b).

Step 6: Graph the function and its three Taylor approximations together. Discuss the graphs in relation to the information discovered in Steps 4 and 5.

53. $f(x) = \dfrac{1}{\sqrt{1 + x}}, \quad |x| \le \dfrac{3}{4}$

54. $f(x) = (1 + x)^{3/2}, \quad -\dfrac{1}{2} \le x \le 2$

55. $f(x) = \dfrac{x}{x^2 + 1}, \quad |x| \le 2$

56. $f(x) = (\cos x)(\sin 2x), \quad |x| \le 2$

57. $f(x) = e^{-x} \cos 2x, \quad |x| \le 1$

58. $f(x) = e^{x/3} \sin 2x, \quad |x| \le 2$

10.10 The Binomial Series and Applications of Taylor Series

We can use Taylor series to solve problems that would otherwise be intractable. For example, many functions have antiderivatives that cannot be expressed using familiar functions. In this section we show how to evaluate integrals of such functions by giving them as Taylor series. We also show how to use Taylor series to evaluate limits that lead to indeterminate forms and how Taylor series can be used to extend the exponential function from real to complex numbers. We begin with a discussion of the binomial series, which comes from the Taylor series of the function $f(x) = (1 + x)^m$, and conclude the section with Table 10.1, which lists some commonly used Taylor series.

Exercises 10.10

Binomial Series

Find the first four terms of the binomial series for the functions in Exercises 1–10.

1. $(1 + x)^{1/2}$
2. $(1 + x)^{1/3}$
3. $(1 - x)^{-3}$
4. $(1 - 2x)^{1/2}$
5. $\left(1 + \frac{x}{2}\right)^{-2}$
6. $\left(1 - \frac{x}{3}\right)^{4}$
7. $(1 + x^3)^{-1/2}$
8. $(1 + x^2)^{-1/3}$
9. $\left(1 + \frac{1}{x}\right)^{1/2}$
10. $\frac{x}{\sqrt[3]{1 + x}}$

Find the binomial series for the functions in Exercises 11–14.

11. $(1 + x)^4$
12. $(1 + x^2)^3$
13. $(1 - 2x)^3$
14. $\left(1 - \frac{x}{2}\right)^{4}$

Approximations and Nonelementary Integrals

T In Exercises 15–18, use series to estimate the integrals' values with an error of magnitude less than 10^{-5}. (The answer section gives the integrals' values rounded to seven decimal places.)

15. $\int_0^{0.6} \sin x^2\, dx$
16. $\int_0^{0.4} \frac{e^{-x} - 1}{x}\, dx$
17. $\int_0^{0.5} \frac{1}{\sqrt{1 + x^4}}\, dx$
18. $\int_0^{0.35} \sqrt[3]{1 + x^2}\, dx$

T Use series to approximate the values of the integrals in Exercises 19–22 with an error of magnitude less than 10^{-8}.

19. $\int_0^{0.1} \frac{\sin x}{x}\, dx$
20. $\int_0^{0.1} e^{-x^2}\, dx$
21. $\int_0^{0.1} \sqrt{1 + x^4}\, dx$
22. $\int_0^{1} \frac{1 - \cos x}{x^2}\, dx$

23. Estimate the error if $\cos t^2$ is approximated by $1 - \frac{t^4}{2} + \frac{t^8}{4!}$ in the integral $\int_0^1 \cos t^2\, dt$.

24. Estimate the error if $\cos \sqrt{t}$ is approximated by $1 - \frac{t}{2} + \frac{t^2}{4!} - \frac{t^3}{6!}$ in the integral $\int_0^1 \cos \sqrt{t}\, dt$.

In Exercises 25–28, find a polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

25. $F(x) = \int_0^x \sin t^2\, dt, \quad [0, 1]$
26. $F(x) = \int_0^x t^2 e^{-t^2}\, dt, \quad [0, 1]$
27. $F(x) = \int_0^x \tan^{-1} t\, dt$, **(a)** $[0, 0.5]$ **(b)** $[0, 1]$
28. $F(x) = \int_0^x \frac{\ln(1 + t)}{t}\, dt$, **(a)** $[0, 0.5]$ **(b)** $[0, 1]$

Indeterminate Forms

Use series to evaluate the limits in Exercises 29–40.

29. $\lim_{x\to 0} \frac{e^x - (1 + x)}{x^2}$
30. $\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$
31. $\lim_{t\to 0} \frac{1 - \cos t - (t^2/2)}{t^4}$
32. $\lim_{\theta\to 0} \frac{\sin\theta - \theta + (\theta^3/6)}{\theta^5}$
33. $\lim_{y\to 0} \frac{y - \tan^{-1} y}{y^3}$
34. $\lim_{y\to 0} \frac{\tan^{-1} y - \sin y}{y^3 \cos y}$
35. $\lim_{x\to\infty} x^2\left(e^{-1/x^2} - 1\right)$
36. $\lim_{x\to\infty} (x + 1)\sin\frac{1}{x + 1}$
37. $\lim_{x\to 0} \frac{\ln(1 + x^2)}{1 - \cos x}$
38. $\lim_{x\to 2} \frac{x^2 - 4}{\ln(x - 1)}$
39. $\lim_{x\to 0} \frac{\sin 3x^2}{1 - \cos 2x}$
40. $\lim_{x\to 0} \frac{\ln(1 + x^3)}{x \cdot \sin x^2}$

Using Table 10.1

In Exercises 41–52, use Table 10.1 to find the sum of each series.

41. $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$
42. $\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \left(\frac{1}{4}\right)^6 + \cdots$
43. $1 - \frac{3^2}{4^2 \cdot 2!} + \frac{3^4}{4^4 \cdot 4!} - \frac{3^6}{4^6 \cdot 6!} + \cdots$
44. $\frac{1}{2} - \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} - \frac{1}{4 \cdot 2^4} + \cdots$
45. $\frac{\pi}{3} - \frac{\pi^3}{3^3 \cdot 3!} + \frac{\pi^5}{3^5 \cdot 5!} - \frac{\pi^7}{3^7 \cdot 7!} + \cdots$
46. $\frac{2}{3} - \frac{2^3}{3^3 \cdot 3} + \frac{2^5}{3^5 \cdot 5} - \frac{2^7}{3^7 \cdot 7} + \cdots$
47. $x^3 + x^4 + x^5 + x^6 + \cdots$
48. $1 - \frac{3^2x^2}{2!} + \frac{3^4x^4}{4!} - \frac{3^6x^6}{6!} + \cdots$
49. $x^3 - x^5 + x^7 - x^9 + x^{11} - \cdots$
50. $x^2 - 2x^3 + \frac{2^2x^4}{2!} - \frac{2^3x^5}{3!} + \frac{2^4x^6}{4!} - \cdots$
51. $-1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots$
52. $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \frac{x^4}{5} + \cdots$

Theory and Examples

53. Replace x by $-x$ in the Taylor series for $\ln(1 + x)$ to obtain a series for $\ln(1 - x)$. Then subtract this from the Taylor series for $\ln(1 + x)$ to show that for $|x| < 1$,

$$\ln\frac{1 + x}{1 - x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

54. How many terms of the Taylor series for $\ln(1 + x)$ should you add to be sure of calculating $\ln(1.1)$ with an error of magnitude less than 10^{-8}? Give reasons for your answer.

55. According to the Alternating Series Estimation Theorem, how many terms of the Taylor series for $\tan^{-1} 1$ would you have to add to be sure of finding $\pi/4$ with an error of magnitude less than 10^{-3}? Give reasons for your answer.

56. Show that the Taylor series for $f(x) = \tan^{-1} x$ diverges for $|x| > 1$.

T **57. Estimating Pi** About how many terms of the Taylor series for $\tan^{-1} x$ would you have to use to evaluate each term on the right-hand side of the equation

$$\pi = 48 \tan^{-1}\frac{1}{18} + 32 \tan^{-1}\frac{1}{57} - 20 \tan^{-1}\frac{1}{239}$$

with an error of magnitude less than 10^{-6}? In contrast, the convergence of $\sum_{n=1}^{\infty}(1/n^2)$ to $\pi^2/6$ is so slow that even 50 terms will not yield two-place accuracy.

58. Use the following steps to prove that the binomial series in Equation (1) converges to $(1 + x)^m$.

a. Differentiate the series

$$f(x) = 1 + \sum_{k=1}^{\infty}\binom{m}{k}x^k$$

to show that

$$f'(x) = \frac{mf(x)}{1 + x}, \quad -1 < x < 1.$$

b. Define $g(x) = (1 + x)^{-m} f(x)$ and show that $g'(x) = 0$.

c. From part (b), show that

$$f(x) = (1 + x)^m.$$

59. a. Use the binomial series and the fact that

$$\frac{d}{dx}\sin^{-1} x = (1 - x^2)^{-1/2}$$

to generate the first four nonzero terms of the Taylor series for $\sin^{-1} x$. What is the radius of convergence?

b. Series for $\cos^{-1} x$ Use your result in part (a) to find the first five nonzero terms of the Taylor series for $\cos^{-1} x$.

60. a. Series for $\sinh^{-1} x$ Find the first four nonzero terms of the Taylor series for

$$\sinh^{-1} x = \int_0^x \frac{dt}{\sqrt{1 + t^2}}.$$

T **b.** Use the first *three* terms of the series in part (a) to estimate $\sinh^{-1} 0.25$. Give an upper bound for the magnitude of the estimation error.

61. Obtain the Taylor series for $1/(1 + x)^2$ from the series for $-1/(1 + x)$.

62. Use the Taylor series for $1/(1 - x^2)$ to obtain a series for $2x/(1 - x^2)^2$.

T **63. Estimating Pi** The English mathematician Wallis discovered the formula

$$\frac{\pi}{4} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot \cdots}.$$

Find π to two decimal places with this formula.

64. The complete elliptic integral of the first kind is the integral

$$K = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}},$$

where $0 < k < 1$ is constant.

a. Show that the first four terms of the binomial series for $1/\sqrt{1 - x}$ are

$$(1 - x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots.$$

b. From part (a) and the reduction integral Formula 67 at the back of the book, show that

$$K = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \cdots\right].$$

65. Series for $\sin^{-1} x$ Integrate the binomial series for $(1 - x^2)^{-1/2}$ to show that for $|x| < 1$,

$$\sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)} \frac{x^{2n+1}}{2n + 1}.$$

66. Series for $\tan^{-1} x$ for $|x| > 1$ Derive the series

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad x > 1$$

$$\tan^{-1} x = -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots, \quad x < -1,$$

by integrating the series

$$\frac{1}{1 + t^2} = \frac{1}{t^2} \cdot \frac{1}{1 + (1/t^2)} = \frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \cdots$$

in the first case from x to ∞ and in the second case from $-\infty$ to x.

Euler's Identity

67. Use Equation (4) to write the following powers of e in the form $a + bi$.

a. $e^{-i\pi}$ **b.** $e^{i\pi/4}$ **c.** $e^{-i\pi/2}$

68. Use Equation (4) to show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

69. Establish the equations in Exercise 68 by combining the formal Taylor series for $e^{i\theta}$ and $e^{-i\theta}$.

70. Show that

a. $\cosh i\theta = \cos\theta$, **b.** $\sinh i\theta = i\sin\theta$.

71. By multiplying the Taylor series for e^x and $\sin x$, find the terms through x^5 of the Taylor series for $e^x \sin x$. This series is the imaginary part of the series for

$$e^x \cdot e^{ix} = e^{(1+i)x}.$$

Use this fact to check your answer. For what values of x should the series for $e^x \sin x$ converge?

72. When a and b are real, we define $e^{(a+ib)x}$ with the equation

$$e^{(a+ib)x} = e^{ax} \cdot e^{ibx} = e^{ax}(\cos bx + i\sin bx).$$

Differentiate the right-hand side of this equation to show that

$$\frac{d}{dx}e^{(a+ib)x} = (a + ib)e^{(a+ib)x}.$$

Thus the familiar rule $(d/dx)e^{kx} = ke^{kx}$ holds for k complex as well as real.

73. Use the definition of $e^{i\theta}$ to show that for any real numbers θ, θ_1, and θ_2,

a. $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, **b.** $e^{-i\theta} = 1/e^{i\theta}$.

74. Two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$. Use this fact to evaluate

$$\int e^{ax}\cos bx\,dx \quad \text{and} \quad \int e^{ax}\sin bx\,dx$$

from

$$\int e^{(a+ib)x}\,dx = \frac{a - ib}{a^2 + b^2}e^{(a+ib)x} + C,$$

where $C = C_1 + iC_2$ is a complex constant of integration.

Chapter 10 Questions to Guide Your Review

1. What is an infinite sequence? What does it mean for such a sequence to converge? To diverge? Give examples.
2. What is a monotonic sequence? Under what circumstances does such a sequence have a limit? Give examples.
3. What theorems are available for calculating limits of sequences? Give examples.
4. What theorem sometimes enables us to use l'Hôpital's Rule to calculate the limit of a sequence? Give an example.
5. What are the six commonly occurring limits in Theorem 5 that arise frequently when you work with sequences and series?
6. What is an infinite series? What does it mean for such a series to converge? To diverge? Give examples.
7. What is a geometric series? When does such a series converge? Diverge? When it does converge, what is its sum? Give examples.
8. Besides geometric series, what other convergent and divergent series do you know?
9. What is the nth-Term Test for Divergence? What is the idea behind the test?
10. What can be said about term-by-term sums and differences of convergent series? About constant multiples of convergent and divergent series?
11. What happens if you add a finite number of terms to a convergent series? A divergent series? What happens if you delete a finite number of terms from a convergent series? A divergent series?
12. How do you reindex a series? Why might you want to do this?
13. Under what circumstances will an infinite series of nonnegative terms converge? Diverge? Why study series of nonnegative terms?
14. What is the Integral Test? What is the reasoning behind it? Give an example of its use.
15. When do p-series converge? Diverge? How do you know? Give examples of convergent and divergent p-series.
16. What are the Direct Comparison Test and the Limit Comparison Test? What is the reasoning behind these tests? Give examples of their use.
17. What are the Ratio and Root Tests? Do they always give you the information you need to determine convergence or divergence? Give examples.
18. What is absolute convergence? Conditional convergence? How are the two related?
19. What is an alternating series? What theorem is available for determining the convergence of such a series?
20. How can you estimate the error involved in approximating the sum of an alternating series with one of the series' partial sums? What is the reasoning behind the estimate?
21. What do you know about rearranging the terms of an absolutely convergent series? Of a conditionally convergent series?
22. What is a power series? How do you test a power series for convergence? What are the possible outcomes?
23. What are the basic facts about
    a. sums, differences, and products of power series?
    b. substitution of a function for x in a power series?
    c. term-by-term differentiation of power series?
    d. term-by-term integration of power series?
    Give examples.
24. What is the Taylor series generated by a function $f(x)$ at a point $x = a$? What information do you need about f to construct the series? Give an example.
25. What is a Maclaurin series?
26. Does a Taylor series always converge to its generating function? Explain.
27. What are Taylor polynomials? Of what use are they?
28. What is Taylor's formula? What does it say about the errors involved in using Taylor polynomials to approximate functions? In particular, what does Taylor's formula say about the error in a linearization? A quadratic approximation?
29. What is the binomial series? On what interval does it converge? How is it used?
30. How can you sometimes use power series to estimate the values of nonelementary definite integrals? To find limits?
31. What are the Taylor series for $1/(1 - x)$, $1/(1 + x)$, e^x, $\sin x$, $\cos x$, $\ln(1 + x)$, and $\tan^{-1}x$? How do you estimate the errors involved in replacing these series with their partial sums?