Math 21C
Kouba
Discussion Sheet 4

1.) Use what you know about converging geometric series to write each power series as an ordinary function.

a.) $\displaystyle\sum_{n=2}^{\infty} \frac{x^n}{3^n}$ b.) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}(x+3)^{n+1}}{5^n}$

c.) $x^2 - x^{5/2} + x^3 - x^{7/2} + x^4 - x^{9/2} + \cdots$ d.) $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n$

2.) Recall that if $y = f(x)$ is a function and

$$a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots = \sum_{n=0}^{\infty} a_n(x-a)^n$$

is the Taylor Series (or Maclaurin series if $a = 0$) centered at $x = a$ for $y = f(x)$, then $a_n = \dfrac{f^{(n)}(a)}{n!}$. Use this formula to compute the first four nonzero terms and the general formula for the Taylor series expansion for each function about the given value of a .

a.) $f(x) = e^x$ centered at $x = 0$ b.) $f(x) = e^x$ centered at $x = \ln 2$

c.) $f(x) = \dfrac{1}{1-x}$ centered at $x = 0$ d.) $f(x) = \sin x$ centered at $x = 0$

e.) $f(x) = \dfrac{1}{x}$ centered at $x = 1$ f.) $f(x) = \sqrt{x+5}$ centered at $x = -1$

3.) Use the suggested method to find the first four nonzero terms of the Maclaurin series for each function.

a.) $f(x) = \dfrac{1}{1+x^2}$ (Substitute $-x^2$ into the Maclaurin series for $\dfrac{1}{1-x}$.)

b.) $f(x) = x^3 e^{-3x}$ (Substitute $-3x$ into the Maclaurin series for e^x and then multiply by x^3 .)

c.) $f(x) = \dfrac{e^x}{1-x} = e^x \dfrac{1}{1-x}$ (Multiply the Maclaurin series for e^x and $\dfrac{1}{1-x}$ term by term and then group like powers of x .)

d.) $f(x) = \dfrac{e^x}{1-x}$ (Use polynomial division. Divide the Maclaurin series for e^x by $1-x$.)

e.) $f(x) = 3x^2 \cos(x^3)$ (Substitute x^3 into the Maclaurin series for $\sin x$ then differentiate term by term.)

f.) $f(x) = \arctan x$ (Integrate the Maclaurin series for $\dfrac{1}{1+t^2}$ from 0 to x.)

4.) The Maclaurin series for $f(x) = \frac{1}{1+x}$ is $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

a.) Show that $f(x) = \frac{1}{1+x}$ and $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ have the same value at $x = 0$.

b.) Show that $f(x) = \frac{1}{1+x}$ and $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ have the same value at $x = 1/2$.

c.) Show that $f(x) = \frac{1}{1+x}$ and $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ do not have the same value at $x = 1$.

d.) For what x-values is $f(x) = \frac{1}{1+x}$ defined ?

e.) For what x-values is the Maclaurin series $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ defined ?

NOTE : It can be shown that $f(x) = \frac{1}{1+x}$ and its Maclaurin series $1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ are equal on the interval $(-1, 1)$.

5.) Determine (Use shortcuts.) the third-degree Taylor polynomial, $P_3(x;0)$, for the function $f(x) = \frac{x}{1+x}$. Use $\int_0^1 P_3(x;0)\,dx$ to estimate the value of $\int_0^1 \frac{x}{1+x}\,dx$. Now evaluate $\int_0^1 \frac{x}{1+x}\,dx$ directly to see how good the estimate is.

6.) The following definite integral cannot be evaluated using the Fundamental Theorem of Calculus. Use the Maclaurin series for $\cos x$ and the absolute error $|R_n|$ for an alternating series to estimate the value of this integral with error at most 0.0001 : $\int_0^1 \cos(x^2)\,dx$

7.) Write each Maclaurin series as an ordinary function.

a.) $(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \frac{(3x)^9}{9!} - \cdots$ (HINT: Use $\sin x$.)

b.) $x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$ (HINT: Factor.)

c.) $\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \frac{x^4}{6!} + \cdots$ (HINT: Use e^x.)

d.) $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \cdots$ (Challenging)

8.) Use any method to find the given Taylor polynomial for each function. Then estimate the Absolute Taylor Error on the indicated interval.

a.) $f(x) = e^{-2x}$, $P_3(x;0)$, for $[-1/2, 1/3]$

b.) $f(x) = \sin 2x$, $P_5(x;0)$, for $[0, 3/4]$

c.) $f(x) = \frac{x}{1-x}$, $P_4(x;0)$, for $[-1/3, 0]$

9.) What should n be so that the nth-degree Taylor Polynomial $P_n(x;a)$ estimates the value of the given function on the indicated interval with Absolute Taylor Error at most 0.00001 ?

a.) $f(x) = e^{-x}$ for $a = 1$ and $[0, 1]$

b.) $f(x) = \dfrac{x+3}{x+1}$ for $a = 0$ and $[0, 1/2]$

10.) (Challenging) Use shortcuts to find the first three nonzero terms in the Taylor Series centered at $x = -1$ for $f(x) = \dfrac{x}{3-x}$.

"In mathematics you don't understand things. You just get used to them." – Johann von Neumann