

Math 21C

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The Limit of Sequence $\left\{ \frac{x^n}{n!} \right\}$

Example : Find the limit of the sequence $\left\{ \frac{3^n}{n!} \right\}$. Start by writing out the first six terms :

$$\begin{aligned} n : & \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\ \frac{3^n}{n!} : & \quad \frac{3}{1}, \frac{3 \cdot 3}{2 \cdot 1}, \frac{3 \cdot 3 \cdot 3}{3 \cdot 2 \cdot 1}, \frac{3 \cdot 3 \cdot 3 \cdot 3}{4 \cdot 3 \cdot 2 \cdot 1}, \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \dots \\ & = 3, \frac{9}{2}, \frac{9}{2}, \left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \left(\frac{3}{5}\right)\left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \left(\frac{3}{6}\right)\left(\frac{3}{5}\right)\left(\frac{3}{4}\right)\left(\frac{9}{2}\right), \dots \\ & \leq 3, \frac{9}{2}, \left(\frac{3}{4}\right)^0\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^1\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^2\left(\frac{9}{2}\right), \left(\frac{3}{4}\right)^3\left(\frac{9}{2}\right), \dots \end{aligned}$$

Thus, for $n \geq 3$ it follows that

$$0 \leq \frac{3^n}{n!} \leq \left(\frac{3}{4}\right)^{n-3} \left(\frac{9}{2}\right)$$

Since $\lim_{n \rightarrow \infty} \left(\frac{3}{4}\right)^{n-3} \left(\frac{9}{2}\right) = (0)\left(\frac{9}{2}\right) = 0$, it

follows from the Sandwich Theorem that $\lim_{n \rightarrow \infty} \frac{3^n}{n!} = 0$.

Rule : Let x be any real number. Then

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0.$$