

Section 11.8

Taylor polynomial of degree n :

$$P_n(x; a) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \dots + a_n(x-a)^n, \quad \text{where}$$

$$a_n = \frac{f^{(n)}(a)}{n!} \quad \text{for } n=0, 1, 2, 3, \dots$$

1.) $f(x) = \ln x$, $a = 1$:

$$f'(x) = \frac{1}{x}, \quad f''(x) = \frac{-1}{x^2}, \quad f'''(x) = \frac{2}{x^3}, \quad \text{so}$$

$$a_0 = \frac{f(1)}{0!} = \frac{\ln 1}{1} = \frac{0}{1} = 0,$$

$$a_1 = \frac{f'(1)}{1!} = \frac{1}{1} = 1, \quad a_2 = \frac{f''(1)}{2!} = \frac{-1}{2},$$

$$a_3 = \frac{f'''(1)}{3!} = \frac{2}{6} = \frac{1}{3}; \quad \text{then}$$

$$P_0(x; 1) = a_0 = 0$$

$$P_1(x; 1) = a_0 + a_1(x-1) = 1 \cdot (x-1) = x-1,$$

$$P_2(x; 1) = a_0 + a_1(x-1) + a_2(x-1)^2 \\ = (x-1) + \frac{-1}{2}(x-1)^2,$$

$$P_3(x; 1) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 \\ = (x-1) + \frac{-1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

4.) $f(x) = \frac{1}{x+2} = (x+2)^{-1}$, $a = 0$:

$$f'(x) = -(x+2)^{-2}, \quad f''(x) = 2(x+2)^{-3},$$

$$f'''(x) = -6(x+2)^{-4}, \quad \text{so}$$

$$a_0 = \frac{f(0)}{0!} = \frac{1/2}{1} = \frac{1}{2}, \quad a_1 = \frac{f'(0)}{1!} = \frac{-1/4}{1} = -\frac{1}{4},$$

$$a_2 = \frac{f''(0)}{2!} = \frac{2 \cdot \frac{1}{8}}{2} = \frac{1}{8},$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{-6 \cdot \frac{1}{16}}{6} = -\frac{1}{16}; \text{ then}$$

$$P_0(x; 0) = a_0 = \frac{1}{2},$$

$$P_1(x; 0) = a_0 + a_1 x = \frac{1}{2} - \frac{1}{4}x,$$

$$P_2(x; 0) = a_0 + a_1 x + a_2 x^2 \\ = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2,$$

$$P_3(x; 0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ = \frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3$$

6.) $f(x) = \cos x$, $a = \frac{\pi}{4}$:

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x,$$

$$\text{so } a_0 = \frac{f(\frac{\pi}{4})}{0!} = \frac{\cos \frac{\pi}{4}}{1} = \frac{\sqrt{2}}{2},$$

$$a_1 = \frac{f'(\frac{\pi}{4})}{1!} = \frac{-\sin \frac{\pi}{4}}{1} = -\frac{\sqrt{2}}{2},$$

$$a_2 = \frac{f''(\frac{\pi}{4})}{2!} = \frac{-\cos \frac{\pi}{4}}{2} = \frac{-\frac{\sqrt{2}}{2}}{2} = -\frac{\sqrt{2}}{4},$$

$$a_3 = \frac{f'''(\frac{\pi}{4})}{3!} = \frac{\sin \frac{\pi}{4}}{6} = \frac{\frac{\sqrt{2}}{2}}{6} = \frac{\sqrt{2}}{12}; \text{ then}$$

$$P_0(x; \frac{\pi}{4}) = a_0 = \frac{\sqrt{2}}{2},$$

$$P_1(x; \frac{\pi}{4}) = a_0 + a_1 (x - \frac{\pi}{4}) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} (x - \frac{\pi}{4}),$$

$$P_2(x; \frac{\pi}{4}) = a_0 + a_1 (x - \frac{\pi}{4}) + a_2 (x - \frac{\pi}{4})^2$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4} \left(x - \frac{\pi}{4}\right)^2,$$

$$P_3(x; \frac{\pi}{4}) = a_0 + a_1 \left(x - \frac{\pi}{4}\right) + a_2 \left(x - \frac{\pi}{4}\right)^2 + a_3 \left(x - \frac{\pi}{4}\right)^3$$

$$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4} \left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{12} \left(x - \frac{\pi}{4}\right)^3$$

8.) $f(x) = \sqrt{x+4}$, $a=0$:

$$f'(x) = \frac{1}{2} (x+4)^{-1/2} \quad f''(x) = -\frac{1}{4} (x+4)^{-3/2}$$

$$f'''(x) = \frac{3}{8} (x+4)^{-5/2}; \quad \text{so}$$

$$a_0 = \frac{f(0)}{0!} = \frac{\sqrt{4}}{1} = 2, \quad a_1 = \frac{f'(0)}{1!} = \frac{\frac{1}{2} \cdot \frac{1}{2}}{1} = \frac{1}{4},$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-\frac{1}{4} \cdot \frac{1}{8}}{2} = -\frac{1}{64},$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{\frac{3}{8} \cdot \frac{1}{32}}{6} = \frac{1}{512}; \quad \text{then}$$

$$P_0(x; 0) = a_0 = 2,$$

$$P_1(x; 0) = a_0 + a_1 x = 2 + \frac{1}{4}x,$$

$$P_2(x; 0) = a_0 + a_1 x + a_2 x^2 = 2 + \frac{1}{4}x - \frac{1}{64}x^2,$$

$$P_3(x; 0) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3$$

9.) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{so}$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots$$

$$= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$10.) e^{\frac{x}{2}} = 1 + \left(\frac{x}{2}\right) + \frac{\left(\frac{x}{2}\right)^2}{2!} + \frac{\left(\frac{x}{2}\right)^3}{3!} + \dots$$

$$= 1 + \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} + \frac{x^3}{2^3 \cdot 3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{2^n \cdot n!}$$

$$11.) \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots$$

$$= 1 - x + x^2 - x^3 + x^4 - \dots = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$13.) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\text{so } \sin 3x = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \dots$$

$$= 3x - \frac{3^3}{3!} x^3 + \frac{3^5}{5!} x^5 - \frac{3^7}{7!} x^7 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{3^{2n+1}}{(2n+1)!} x^{2n+1}$$

$$20.) f(x) = (x+1)^2, \quad a=0:$$

$$f'(x) = 2(x+1), \quad f''(x) = 2, \quad f'''(x) = 0, \dots$$

$$\text{so } a_0 = \frac{f(0)}{0!} = \frac{1^2}{1} = 1, \quad a_1 = \frac{f'(0)}{1!} = \frac{2}{1} = 2,$$

$$a_2 = \frac{f''(0)}{2!} = \frac{2}{2} = 1, \quad a_3 = \frac{f'''(0)}{3!} = \frac{0}{6} = 0,$$

$$a_4 = a_5 = a_6 = \dots = 0; \quad \text{then}$$

$$(x+1)^2 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= 1 + 2x + 1 \cdot x^2 + 0 \cdot x^3 + 0 \cdot x^4 + 0 \cdot x^5 + \dots$$

$$= 1 + 2x + x^2$$

$$23.) f(x) = x^4 + x^2 + 1, \quad a = -2 :$$

$$f'(x) = 4x^3 + 2x, \quad f''(x) = 12x^2 + 2$$

$$f'''(x) = 24x, \quad f^{(4)}(x) = 24, \quad f^{(5)}(x) = 0,$$

$$f^{(6)}(x) = f^{(7)}(x) = f^{(8)}(x) = \dots = 0;$$

$$a_0 = \frac{f(-2)}{0!} = \frac{21}{1} = 21, \quad a_1 = \frac{f'(-2)}{1!} = \frac{-36}{1} = -36,$$

$$a_2 = \frac{f''(-2)}{2!} = \frac{50}{2} = 25, \quad a_3 = \frac{f'''(-2)}{3!} = \frac{-48}{6} = -8,$$

$$a_4 = \frac{f^{(4)}(-2)}{4!} = \frac{24}{24} = 1, \quad a_5 = a_6 = a_7 = \dots = 0;$$

$$\begin{aligned} \text{then } x^4 + x^2 + 1 &= a_0 + a_1(x - (-2)) + a_2(x - (-2))^2 + \dots \\ &= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4 + 0(x+2)^5 + \dots \\ &= 21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4 \end{aligned}$$

$$25.) f(x) = \frac{1}{x^2}, \quad a = 1 :$$

$$f'(x) = -2x^{-3}, \quad f''(x) = 3 \cdot 2x^{-4} = 3!x^{-4},$$

$$f'''(x) = -4 \cdot 3!x^{-5} = -4!x^{-5},$$

$$f^{(4)}(x) = 5 \cdot 4!x^{-6} = 5!x^{-6}$$

$$f^{(n)}(x) = (n+1)!x^{-(n+2)} \quad \text{for } n = 0, 1, 2, 3, \dots;$$

then

$$a_0 = \frac{f(1)}{0!} = \frac{1}{1} = 1, \quad a_1 = \frac{f'(1)}{1!} = \frac{-2}{1} = -2,$$

$$a_2 = \frac{f''(1)}{2!} = \frac{3!}{2!} = 3, \quad a_3 = \frac{f'''(1)}{3!} = \frac{-4!}{3!} = -4,$$

$$a_4 = \frac{f^{(4)}(1)}{4!} = \frac{5!}{4!} = 5, \dots, \quad a_n = (-1)^n \cdot (n+1)$$

for $n = 0, 1, 2, 3, \dots$; then

$$\begin{aligned} \frac{1}{x^2} &= a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + \dots \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot (n+1) \cdot (x-1)^n. \end{aligned}$$

$$\begin{aligned} 26.) \frac{x}{1-x} &= x \cdot \frac{1}{1-x} = x \cdot (1 + x + x^2 + x^3 + \dots) \\ &= x + x^2 + x^3 + x^4 + \dots = \sum_{n=1}^{\infty} x^n \end{aligned}$$

$$\begin{aligned} 27.) f(x) &= e^x, \quad a=2: \\ f'(x) &= e^x, \quad f^{(n)}(x) = e^x \text{ for } n=0, 1, 2, 3, \dots; \\ \text{so } a_0 &= \frac{f(2)}{0!} = \frac{e^2}{1} = e^2, \quad a_1 = \frac{f'(2)}{1!} = \frac{e^2}{1} = e^2, \\ a_2 &= \frac{f''(2)}{2!} = \frac{e^2}{2!}, \quad a_3 = \frac{e^2}{3!}, \dots, \quad a_n = \frac{e^2}{n!} \text{ for } \\ n &= 0, 1, 2, 3, \dots; \text{ then} \\ e^x &= a_0 + a_1(x-2) + a_2(x-2)^2 + a_3(x-2)^3 + \dots \\ &= e^2 + e^2(x-2) + \frac{e^2}{2!}(x-2)^2 + \frac{e^2}{3!}(x-2)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n. \end{aligned}$$

$$\begin{aligned} 33.) f(x) &= \ln(\cos x), \quad a=0: \\ f'(x) &= \frac{1}{\cos x} \cdot -\sin x = -\tan x, \\ f''(x) &= -\sec^2 x; \quad \text{so } a_0 = \frac{f(0)}{0!} = \frac{\ln(\cos 0)}{1} \\ &= \frac{\ln 1}{1} = \frac{0}{1} = 0, \quad a_1 = \frac{f'(0)}{1!} = \frac{-\tan 0}{1} = 0, \end{aligned}$$

$$a_2 = \frac{f''(0)}{2!} = \frac{-\sec^2 0}{2} = \frac{-1}{2}; \text{ then}$$

$$a.) P_1(x; 0) = a_0 + a_1 x = 0 + 0 \cdot x = 0$$

$$b.) P_2(x; 0) = a_0 + a_1 x + a_2 x^2 \\ = 0 + 0 \cdot x - \frac{1}{2} x^2 = -\frac{1}{2} x^2$$

$$35.) f(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}, \quad a=0:$$

$$f'(x) = -\frac{1}{2} (1-x^2)^{-3/2} \cdot (-2x) = \frac{x}{(1-x^2)^{3/2}}$$

$$f''(x) = \frac{(1-x^2)^{3/2}(1) - x \cdot \frac{3}{2} (1-x^2)^{1/2} \cdot (-2x)}{(1-x^2)^3} \\ = \frac{(1-x^2)^{1/2} \cdot [(1-x^2) + 3x^2]}{(1-x^2)^3} = \frac{1-2x^2}{(1-x^2)^{5/2}};$$

$$\text{then } a_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1,$$

$$a_1 = \frac{f'(0)}{1!} = \frac{0}{1} = 0, \quad a_2 = \frac{f''(0)}{2!} = \frac{1}{2}; \text{ then}$$

$$a.) P_1(x; 0) = a_0 + a_1 x = 1 + 0 \cdot x = 1$$

$$b.) P_2(x; 0) = a_0 + a_1 x + a_2 x^2 \\ = 1 + 0 \cdot x + \frac{1}{2} x^2 = 1 + \frac{1}{2} x^2$$