<u>Theorem</u> (Second Derivative Test for $f(x,y)$): Assume f has continuous first and second order partial derivatives and assume (a,b) is a critical point for f. Let

$$D = f_{xx}(a,b) \cdot f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2 .$$

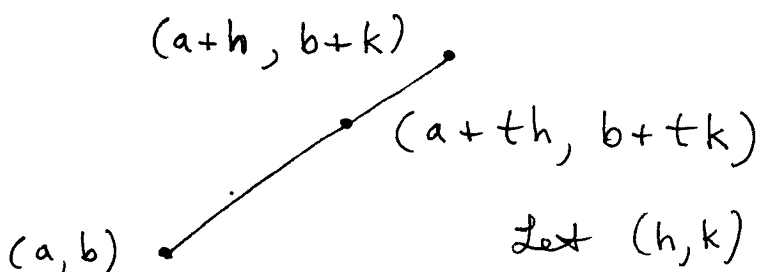
1.) If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) determines a relative minimum value at $(a, b, f(a,b))$.

2.) If $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) determines a relative maximum value at $(a, b, f(a,b))$.

3.) If $D < 0$, then (a,b) determines a saddle point at $(a, b, f(a,b))$.

4.) If $D = 0$, then this test is inconclusive.

<u>proof</u>:



Let (h,k) be such that $(a+h, b+k)$ is near (a,b) and $(a+th, b+tk)$ is some point between

(a,b) and $(a+h, b+k)$, i.e., $0 \le t \le 1$ so that $(a+th, b+tk)$ is on the line segment joining (a,b) and $(a+h, b+k)$.

Define a new function G given by

$$G(t) = f(a+th, b+tk)$$

for $0 \le t \le 1$. By Taylor's formula applied to $G(t)$ on $[0,1]$ we have that

$$(1) \qquad G(1) = G(0) + G'(0) + \tfrac{1}{2} G''(c)$$

where $0 < c < 1$. By the chain rule it follows that

$$(2) \qquad G'(t) = f_X \cdot \frac{dX}{dt} + f_Y \cdot \frac{dY}{dt} = h \cdot f_X + k \cdot f_Y$$

and

$$G''(t) = \frac{\partial}{\partial X}(hf_X + kf_Y) \cdot \frac{dX}{dt} + \frac{\partial}{\partial Y}(hf_X + kf_Y) \cdot \frac{dY}{dt}$$

$$= (hf_{XX} + kf_{YX})h + (hf_{XY} + kf_{YY})k$$

$$(3) \qquad = h^2 f_{XX} + 2hk f_{XY} + k^2 f_{YY},$$

where all partial derivatives are evaluated at the point $(a+th, b+tk)$. Then by (2)

$$G'(0) = h f_X(a,b) + k f_Y(a,b) = h \cdot 0 + k \cdot 0 = 0$$

since (a,b) is a critical point for f, and by (3)

$$G''(c) = A h^2 + 2B hk + C k^2 ,$$

where $A = f_{xx}(a+ch, b+ck)$, $B = f_{xy}(a+ch, b+ck)$, and $C = f_{yy}(a+ch, b+ck)$. Since $G(0) = f(a,b)$ and $G(1) = f(a+h, b+k)$ it follows from (1) that

(4) $$f(a+h, b+k) = f(a,b) + \frac{1}{2}(Ah^2 + 2Bhk + Ck^2) .$$

Let k be fixed and consider the quadratic expression

(5) $$q(h) = g(h,k) = Ah^2 + 2Bhk + Ck^2$$
$$= (A)h^2 + (2Bk)h + (Ck^2) .$$

By the quadratic formula the roots of q are

$$h = \frac{-2Bk \pm \sqrt{4B^2k^2 - 4ACk^2}}{2A} = \frac{-Bk \pm |k|\sqrt{B^2 - AC}}{A} .$$

(i) If $AC - B^2 > 0$ and $A > 0$, then the quadratic in (5) is always positive-valued. It follows from (4) that $f(a+h, b+k) > f(a,b)$, i.e., $f(a,b)$ is a minimum value.

(ii) If $AC - B^2 > 0$ and $A < 0$, then the quadratic in (5) is always negative-valued. It follows from (4) that $f(a+h, b+k) < f(a,b)$, i.e., $f(a,b)$ is a maximum value.

(iii) If $AC - B^2 < 0$, then the quadratic in (5) assumes both positive and negative values. It follows from (4) that $f(a+h, b+k) < f(a,b)$ for some (h,k) and $f(a+h, b+k) > f(a,b)$ for some (h,k), i.e., f has a saddle point at (a,b).

By the continuity of f and its partial derivatives, it follows that

$$\mathcal{D} = f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

and

$$AC - B^2$$

have the same sign, and $f_{xx}(a,b)$ and A have the same sign. Thus, 1.), 2.), and 3.) follow from (i), (ii), and (iii).

Q.E.D.